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Numerical Solution of Ordinary Differential Equations of First Order

Many challenges in science and engineering can be reduced to the task of solving differential equations while satisfying certain predefined conditions. Traditional analytical techniques, which are assumed to be understood by the reader, are suitable for solving only a subset of these equations. However, the differential equations that govern the behavior of physical systems often lack closed-form solutions. Therefore, it is crucial to use numerical methods to solve these problems.

Definition 1. An ordinary differential equation (ODE) of order $n, n \in \mathbb{N}^*$ is any relation of the type

$$f(t, y(t), y'(t), \ldots, y^{(n)}(t)) = 0 \tag{1}$$

which we write in the canonical form as

$$y^{(n)}(t) = f(t, y(t), y'(t), \ldots, y^{(n-1)}(t)) \tag{2}$$

where y is a function of the variable t, and for $i = 1, \ldots, n, y^{(i)}$ is the derivative of y with respect to t of order i.

The general solution of equations (1) and (2) is given by a relation between t and y with a number of constants (equal to the degree of the equation). This relation can be implicit :

$$W(t, y(t), c_1, \ldots, c_n) = 0$$

or explicit

$$y(t) = V(t, c_1, \ldots, c_n).$$

To determine the constants $c_i, i = 1, \ldots, n$, we need initial or boundary conditions on y.

Definition 2. A differential equation is said to be of order 1 if it is of the form : $y'(t) = f(t, y(t))$ with $t \in [a, b]$ and f a function is defined on $[a, b] \times \mathbb{R} \to \mathbb{R}$.

In this chapter, we present some numerical methods that aim to approximate solutions of ordinary differential equations of first order.

1. The Basic Principles of Initial-Value Problems

Definitions and results from ordinary differential equations theory are necessary before delving into methods for approximating solutions of initial-value problems.

1.1. Cauchy problem

The goal is to find a differentiable function $y(t) : I = [a, b] \to \mathbb{R}$ such that

$$(P) \begin{cases} y'(t) = f(t, y(t)), t \in I \\ y(t_0) = y_0(\text{ Initial condition}) \end{cases} \tag{P}$$

1.2. Existence and uniqueness of the solution

Theorem 1. *If $f(t, y)$ is a continuous function on $I \times \mathbb{R}$, then the problem (P) admits a solution. The uniqueness of the solution is guaranteed under one of the following conditions :*

a- $f(t, y)$ satisfies the Lipschitz condition with respect to y, i.e.,

$$\exists L > 0, \forall y_1, y_2 \in \mathbb{R} : |f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

b- The partial derivative $\frac{\partial f}{\partial y}(t, y)$ is continuous and bounded on $I \times \mathbb{R}$.

Example 1.

$$(P_1):\begin{cases} y'(t)=\frac{-y}{t\ln t}+\frac{1}{\ln t}, & t\in[e,5]\\ y(e)=e \end{cases}$$

We have $f(t,y)=\frac{-y}{t\ln t}+\frac{1}{\ln t}$, which is continuous, and $|\frac{\partial f}{\partial y}(t,y(t))|=|\frac{-1}{t\ln t}|\leq\frac{1}{e}$, so $\frac{\partial f}{\partial y}$ is continuous and bounded on $[e,5]\times\mathbb{R}$. Therefore, the problem (P_1) admits a unique solution $y(t)=\frac{t}{\ln t}$.

Example 2.

$$(P_2):\begin{cases} y'(t)=1+t\sin(ty(t)), & t\in[0,2]\\ y(0)=0 \end{cases}$$

We have $f(t,y)=1+t\sin(ty(t))$, which is continuous, and $\frac{\partial f}{\partial y}(t,y(t))=t^2\cos(ty(t))\leq t^2\leq 4$, so $\frac{\partial f}{\partial y}$ is bounded. Therefore, the problem (P_2) admits a unique solution.

Example 3. Consider the following IVP :

$$y^{'}=-\frac{y}{t\ln t}+\frac{1}{\ln t},\quad t\in[e,5],\quad y(e)=e. \tag{3}$$

The function $f(t,y)=-\frac{y}{t\ln t}+\frac{1}{\ln t}$ is continuous on $D=[e,5]\times\mathbb{R}$. Additionally, we have :

$$\left|\frac{\partial f}{\partial y}(t,y)\right|=\left|\frac{-1}{t\ln t}\right|\leq\frac{1}{e}.$$

Therefore, the IVP (3) possesses a unique solution $y(t)=\frac{t}{ln(t)}$.

Example 4. Consider the following IVP :

$$y^{'}=1+t\sin(ty),\quad t\in[0,2],\quad y(0)=0. \tag{4}$$

The function $f(t,y)=1+t\sin(ty)$ is continuous on $D=[0,2]\times\mathbb{R}$,

$$\frac{\partial f}{\partial y}(t,y)=t^2\cos(ty)\leq 4.$$

Hence, the IVP (4) has a unique solution.

2. Picard's Method of Succssive Approximations

Upon integrating the an IVP, we integrate the following integral equation

$$y=y_0+\int_{t_0}^{t}f(t,y)dt. \tag{5}$$

Equation (5), wherein the unknown function y appears under the integral sign, is termed an integral equation. Such an equation can be resolved through the method of successive approximations, wherein the initial approximation to y is acquired by substituting y_0 for y on the right side of Eq. (5). Thus, we express :

$$y_1=y_0+\int_{t_0}^{t}f\left(t,y_0\right)dt.$$

The integral on the right can now be evaluated, yielding y_1, which is then substituted for y in the integrand of Eq. (5), resulting in the second approximation y_2 :

$$y^{(2)}=y_0+\int_{t_0}^{t}f\left(t,y^{(1)}\right)dt.$$

Continuing iteratively, we obtain $y_3,y_4,\ldots,y_{n-1}$ and y_n, where

$$y_n=y_0+\int_{t-0}^{t}f\left(t,y_{n-1}\right)dt. \tag{6}$$

Thus, this method provides a sequence of approximations $y_1,y_2,\ldots,y_n$, and it can be demonstrated that if the function $f(t,y)$ remains bounded in a certain region around the point (t_0,y_0) and if $f(t,y)$ satisfies the Lipschitz condition, then the sequence $y_1,y_2,\ldots$ converges to the solution of Problem (P).

Example 5. Consider the following IVP

$$y' = t + y^2, \quad t \in [0, +\infty[, \quad y(0) = 1.$$

Starting with $y_0 = 1$, the first approximation is

$$y^{(1)} = 1 + \int_0^t (t+1)dt = 1 + t + \frac{1}{2}t^2.$$

The second approximation is

$$\begin{aligned} y^{(2)} &= 1 + \int_0^t \left[t + \left(1 + t + \frac{1}{2}t^2\right)^2 \right] dt \\ &= 1 + t + \frac{3}{2}t^2 + \frac{2}{3}x^3 + \frac{1}{4}t^4 + \frac{1}{20}t^5. \end{aligned}$$

It is evident that as we proceed to higher approximations, the integrations may become increasingly challenging.

3. Numerical Methods

3.1. Euler Method

The Euler method is the simplest numerical method that allows approximating a solution of first-order ordinary differential equations with initial conditions. To numerically solve the Cauchy problem (P), we begin by partitioning the interval $I = [a, b]$, i.e., we choose points $t_0, t_1, \ldots, t_n$ such that $a = t_0 < t_1 < \cdots < t_n = b$, with $t_{i+1} = t_i + h$, $h = \frac{b-a}{n}$ (the step size) with n is the number of evaluated points. The tangent to the curve $y = y(t)$ at $t = t_0$ has the equation :

$$\tilde{y}(t) = y(t_0) + (t - t_0)y'(t_0)$$

where

$$\tilde{y}(t) = y(t_0) + (t - t_0)f(t_0, y(t_0)).$$

At the point $t = t_1$, we get (see Figure 1) :

$$y(t_1) \simeq \tilde{y}(t_1) = y(t_0) + (t_1 - t_0)f(t_0, y(t_0)),$$

by setting $h = t_1 - t_0$, it become

$$y(t_1) \simeq \tilde{y}(t_1) = y(t_0) + hf(t_0, y(t_0)).$$

Let $y_0 = \tilde{y}(t_0), y_1 = \tilde{y}(t_1)$, and then repeat the same procedure in the interval $[t_1, t_2]$, we obtain :

$$y(t_2) \simeq y_2 = y_1 + hf(t_1, y_1).$$

Thus, continuing in this way, we construct the following Euler algorithm :

$$\begin{cases} y_0 = y(t_0), t_0 = a \\ y_{i+1} = y_i + hf(t_i, y_i), i = 1, \ldots, n-1 \end{cases}$$

where $h = \frac{b-a}{n}$, and $t_{i+1} = t_i + h$.

Definition 3. A numerical method that approximates $y(t_i)$ by y_i with an error $e_i = |y(t_i) - y_i|$ with

$$e_i \leq kh^p$$

is said of order p, where k is a constant independent of i and h, and $y(t_i)$ is the exact value of the solution of the Cauchy problem at the point $t_i = t_0 + ih$.

Theorem 2. *Let $f(t, y)$ be a continuous function on $[a, b] \times \mathbb{R}$ and L-Lipschitz continuous with respect to the variable y, and let $y(t) \in \mathcal{C}^2[a, b]$. Then we have*

$$e_i \leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h$$

where $M_2 = \max_{t \in [a,b]} |y''(t)|$, and e_i is the error made at the point (t_i, y_i), i.e., $e_i = |y(t_i) - y_i|$.

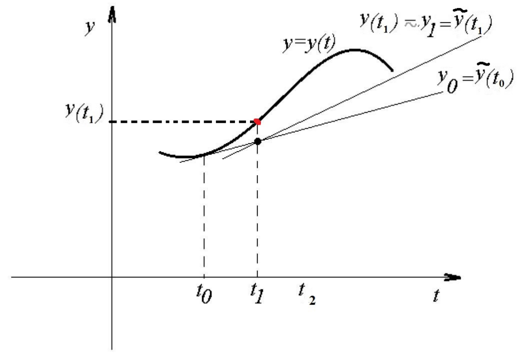

FIGURE 1: Construction of the first iterates of the Euler method.

Remark 1. This result can be expressed in the form $e_i \leq kh$, meaning that the Euler method is of order 1.

Example 6. Consider the following Cauchy problem be given :

$$\begin{cases} y'(t) = ty^{1/3} \\ y(1) = 1 \end{cases}$$

Let's calculate $y(1.01), y(1.02), y(1.03)$ using the Euler method.
We take $y_0 = 1, t_0 = 1$ with $y_{i+1} = y_i + h(t_i y_i^{1/3})$ and $h = 0.01$, the it results that :

$$\begin{aligned} y(1.01) &\simeq y_1 = y_0 + 001 \times t_0 \times y_0^{1/3} = 1 + 0.01 \times 1 \times 1^{1/3} = 1.01. \\ y(1.02) &\simeq y_2 = y_1 + 0.01 \times 1.01 \times (1.01)^{1/3} = 1.0201 \\ y(1.03) &\simeq y_3 = y_2 + 0.01 \times 1.0201 \times (1.0201)^{1/3} = 1.0304. \end{aligned}$$

Example 7. Solve the following Cauchy problem using Euler method with a step size $h = 0.25$.

$$\begin{cases} y'(t) = 2 - ty^2,\ t \in [0, 1] \\ y(0) = 1 \end{cases}$$

The points t_i to evaluate for $h = 0.25$ are $t_0 = 0, t_1 = 0.25, t_2 = 0.5, t_3 = 0.75, t_4 = 1$. Following the same scheme as in the previous example, we obtain :

$$\begin{aligned} y(0.25) &\simeq y_1 = y_0 + 0.25 \times f(t_0, y_0) = 1 + 0.25(2 - 0 \times 1^2) = 1.5 \\ y(0.50) &\simeq y_2 = y_1 + 0.25 \times f(t_1, y_1) = 1.5 + 0.25(2 - 0.25 \times 1.5^2) = 1.8594 \\ y(0.75) &\simeq y_3 = y_2 + 0.25 \times f(t_2, y_2) = 1.859 + 0.25(2 - 0.5 \times 1.859^2) = 1.927 \\ y(1.00) &\simeq y_4 = y_3 + 0.25 \times f(t_3, y_3) = 1.927 + 0.25(2 - 0.75 \times 1.927^2) = 1.7308. \end{aligned}$$

Example 8. Consider the following Cauchy problem :

$$\begin{cases} y'(t) = t + y, t \in [0, 1] \\ y(0) = 1. \end{cases}$$

We want to approximate the solution of this problem at $t = 1$ using Euler method, by subdividing the interval $[0, 1]$ into ten equal parts. Following the same procedure, we obtain the values $\{t_i, y_i\}$ as listed below :

i	0	1	2	3	4	5	6	7	8	9	10
t_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_i	1	1,1	1,22	1,362	1,5282	1,7210	1,9431	2,1974	2,4871	2,8158	3,1874

From this table, we obtain $y(1) \simeq y_{10} = 3.187$. This approximation is quite rough because the exact solution to this problem is given by $y(t) = 2e^t - t - 1$, so the exact value is $y(1) = 3.437$.

3.2. Improved Euler Method

This method is more precise than the previous one ; it consists of replacing, in Euler method, the slope of the tangent at (x_n, y_n) with the corrected value at the midpoint of the interval $[x_n, x_{n+1}]$, whose algorithm is :

$$\begin{cases} y_0 = y(t_0), t_0 = a \\ y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1), i = 1, \ldots, n-1 \\ K_1 = f(x_i, y_i) \end{cases}$$

Example 9. Consider the following Cauchy problem :

$$\begin{cases} y'(t) = y(t) - t + 2, t \in [0, 1] \\ y(0) = 2 \end{cases}$$

Using the Improved Euler method with a step size of $h = 0.1$, we obtain

$$\begin{cases} y_0 = y(0) = 2, h = 0.1 \\ y_1 = y(0.1) = y_0 + hf(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1), \\ K_1 = f(t_0, y_0) = f(0.2) = 4 \\ y(0.1) \simeq y_1 = 2 + \frac{0.1}{2}f(0.05, 2.2) = 2.415. \end{cases}$$

Proceeding the same process, we obtain the results in the following table :

i	0	1	2	3	4	5	6	7	8	9	10
t_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_i	2	2.415	2.8465	3.3111	3.8122	4.3535	4.9388	5.5727	6.2599	7.0059	7.8165

3.3. Second-order Runge-Kutta (Heun's) Method

Runge-Kutta methods approximate the solution with higher accuracy (they generate numerical solutions that are closer to the analytical solutions) than the Euler method. The second-order Runge-Kutta method (RK_2) is an amelioration of the Euler method. Indeed, the Euler method relies on a first-order Taylor expansion. However, it is clear that more efficient methods can be obtained by considering expansions of higher order than 1. Thus, if the function f is sufficiently differentiable, we can write :

$$y_{i+1} = y_i + h \times y'(t_i) + \frac{h^2}{2}y''(t_i)$$

with,

$$y'(t) = f(t, y) \text{ et } y''(t) = \frac{\delta f}{\delta t}(t, y) + f(t, y) \times \frac{\delta f}{\delta y}(t, y).$$

Hence,

$$y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}\left(\frac{\delta f}{\delta t}(t_i, y_i) + f(t_i, y_i) \times \frac{\delta f}{\delta y}(t_i, y_i)\right),$$

since we have : $f(t_i + h, y_i + hf(t_i, y_i)) = f(t_i, y_i) + h\left(\frac{\delta f}{\delta t}(t_i, y_i) + f(t_i, y_i) \times \frac{\delta f}{\delta y}(t_i, y_i)\right)$, it results that

$$y(t_{i+1}) = y(t_i) + \frac{h}{2}f(t_i, y_i) + \frac{h}{2}f(t_i + h, y_i + hf(t_i, y_i)).$$

Thus, we obtain the second-order Runge-Kutta algorithm :

$$(RK_2)\begin{cases} y_0 = y(t_0), t_0 = a \text{ and } h = \frac{b-a}{n} \\ y_{i+1} = y_i + \frac{h}{2}(K_1 + K_2), i = 1, \ldots, n-1 \\ K_1 = f(t_i, y_i) \\ K_2 = f(t_i + h, y_i + hK_1) \end{cases}$$

3.4. Fourth-order Runge–Kutta Method

This is the most accurate and widely used method in practice, with an error of order four. It calculates the value of the function at four intermediate points. Its iterative scheme is given as follows :

$$(RK_4)\begin{cases} y_0 = y(t_0), t_0 = a \text{ and } h = \frac{b-a}{n} \\ y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \dots, n-1 \\ K_1 = f(t_i, y_i) \\ K_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1) \\ K_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_2) \\ K_4 = f(t_i + h, y_i + hK_3) \end{cases}$$

Note that the number of terms retained in the Taylor series defines the order of the Runge-Kutta method. The Runge-Kutta method of order 4 truncates the Taylor series at the term $O(h^4)$.

Example 10. Let the following Cauchy problem :

$$\begin{cases} y'(t) = y - \frac{2t}{y}, t \in [0,1] \\ y(0) = 1. \end{cases}$$

The exact solution of this problem is : $y(t) = \sqrt{2t+1}$.
- Compute an approximate value of $y(0.2)$ using the RK_2 and RK_4 methods with a step size $h = 0.2$.
- Evaluate the obtained results by comparing them with the exact solution.

Runge-Kutta Method of Order 2 :

$$(RK_2)\begin{cases} y_0 = y(0) = 1, h = 0.2 \\ y_1 = y(0.2) = y_0 + \frac{h}{2}(K_1 + K_2), \\ K_1 = f(t_0, y_0) = f(0,1) = 1 \\ K_2 = f(t_0 + h, y_0 + hK_1) = f(0.2, 1.2) = 0.866 \\ y_1 = y(0.2) = 1 + \frac{0.2}{2}(1 + 0.866) = 1.1866. \end{cases}$$

$e_{RK_2} = |\sqrt{2 \times (0.2) + 1} - 1.1866| = 3.450709 \times 10^{-3}$.

Runge-Kutta Method of Order 4 :

$$(RK_4)\begin{cases} y_0 = y(0) = 1, h = 0.2 \\ y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\ K_1 = f(t_0, y_0) = 1 \\ K_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1) = f(0.1, 1.1) = 0.918182 \\ K_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2) = f(0.1, 1.091818) = 0.908637 \\ K_4 = f(t_0 + h, y_i + hK_3) = f(0.2, 1.181727) = 0.843239 \\ y_1 = 1 + \frac{0.2}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.1832292 \end{cases}$$

$e_{RK_4} = |\sqrt{2 \times (0.2) + 1} - 1.1832292| = 1.32 \times 10^{-5}$. Hence $e_{RK_4} \ll e_{RK_2}$.

Example 11. Give an approximate solution of the following Cauchy problem using the RK_4 method with a step size of $h = 0.25$.

$$\begin{cases} y'(t) = 2 - ty^2, t \in [0,1] \\ y(0) = 1. \end{cases}$$

For the first step, we have

$$(RK_4)\begin{cases} y_0 = y(0) = 1, h = 0.25 \\ y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \dots, n-1 \\ K_1 = f(t_0, y_0) = 2 \\ K_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1) = 1.8047 \\ K_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2) = 1.8122 \\ K_4 = f(t_0 + h, y_i + hK_3) = 1.4722 \\ y_1 = 1 + \frac{0.25}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 1.4461 \end{cases}$$

By proceeding the same process as in step 1, we obtain : $y_2 = 1.7028, y_3 = 1.7317$ et $y_4 = 1.6147$.