# Chapter 1

# Special functions

In this chapter, we present some definitions of special functions such as: the Gamma function, the Beta function and the Mittag-Leffler function [5],[6].

## 1.1 Gamma function

**Definition 1.1.1** *We call the gamma function, the function defined by*

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt, z \in \mathbb{C}, \mathrm{Re}(z) > 0,$$

*with* $t^{z-1} = e^{(z-1)\ln t}$.

**Exemple 1.1.1**

$$1)\ \Gamma(1) = \int_0^{+\infty} e^{-t} dt = 1.$$

$$\begin{aligned} 2)\ \Gamma\left(\frac{1}{2}\right) &= \int_0^{+\infty} t^{\frac{1}{2}-1} e^{-t} dt \\ &= \int_0^{+\infty} t^{-\frac{1}{2}} e^{-t} dt \\ &= 2\int_0^{+\infty} e^{-\tau^2} d\tau \\ &= \sqrt{\pi}. \end{aligned}$$

*Using variable change* $t = \tau^2$.

**Lemma 1.1.1** *The gamma function is a function of class $C^{\infty}$ on $\mathbb{R}_{+}^{*}$ (resp. holomorphic on the half plane $z \in \mathbb{C}, \mathrm{Re}(z) > 0$) and $\forall k \in \mathbb{N}^{*}, \forall z \in \mathbb{R}_{+}^{*}$ (resp. $z \in \mathbb{C}, \mathrm{Re}(z) > 0$)*

$$\Gamma^{(k)}(z) = \int_{0}^{+\infty} (\ln t)^{k} t^{z-1} e^{-t} dt.$$

**Lemma 1.1.2** *For all $z \in \mathbb{C}, \mathrm{Re}(z) > 0, n \in \mathbb{N}$, we have*

$$1)\ \Gamma(z+1) = z\Gamma(z).$$

$$2)\ \Gamma(n) = (n-1)!$$

$$3)\ \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^{n} n!}.$$

**Proof.** 1) Let us represent $\Gamma(z+1)$ by the Euler integral and integrate by parts, we get

$$\begin{aligned} \Gamma(z+1) &= \int_{0}^{+\infty} t^{z} e^{-t} dt \\ &= \left[-t^{z} e^{-t}\right]_{0}^{+\infty} + z \int_{0}^{+\infty} t^{z-1} e^{-t} dt \\ &= z\Gamma(z). \end{aligned}$$

2) We just need to apply 1 to $z = n - 1$.

3) We will prove formula $\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{4^{n} n!}$, by induction on $n \in \mathbb{N}$.

- For $n = 0$, we have $\Gamma\left(0 + \frac{1}{2}\right) = \frac{(2 \times 0)!\sqrt{\pi}}{4^{0} 0!} = \sqrt{\pi}$.

- Suppose the formula holds for $(n-1)$ and consider $n$, i.e. suppose

$$\Gamma\left((n-1) + \frac{1}{2}\right) = \frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!},$$

is verified.

So

$$\begin{aligned} \Gamma\left(n + \frac{1}{2}\right) &= \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right) \\ &= \left(n - \frac{1}{2}\right) \frac{(2(n-1))!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\ &= \left(\frac{2n-1}{2}\right) \frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\ &= \frac{2n}{2n} \frac{(2n-1)}{2} \frac{(2n-2)!\sqrt{\pi}}{4^{(n-1)}(n-1)!} \\ &= \frac{(2n)!\sqrt{\pi}}{4^{n} n!}. \end{aligned}$$

So the formula is verified for $n$. ■

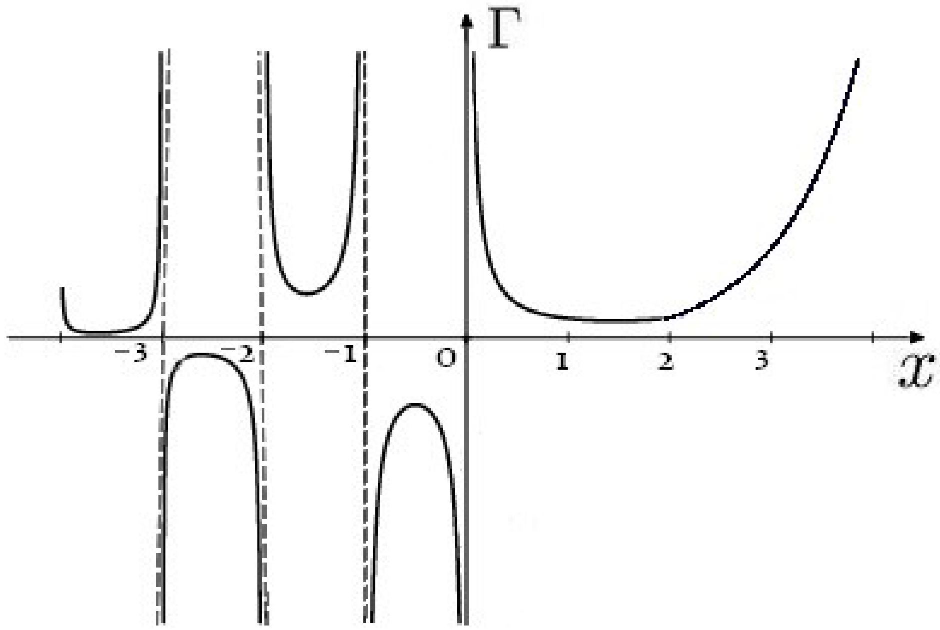


Figure 1.1.1 : The graph of the gamma function

**Remark 1.1.1** *The determination of the gamma function for non-integer negative values by formula* $\Gamma(z) = \frac{\Gamma(z+1)}{z}$*, and the transition from one interval to another* $(-1, 0), (-2, -1), (-3, -2), ...$ *The gamma function does not exist for negative integer values.*

**Exemple 1.1.2**

$$1)\ \Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi}.$$

$$2)\ \Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = \frac{4\sqrt{\pi}}{3}.$$

**Proposition 1.1.1** *For all $p > 0$, we have*

$$\Gamma(p) = \lim_{n\to+\infty} \frac{n!n^p}{p(p+1)(p+2)...(p+n)}.$$

**Proof.** Consider the function

$$f(n,p) = \int_0^n \left(1-\frac{x}{n}\right)^n x^{p-1}dx.$$

We can easily see that

$$\lim_{n\to+\infty} f(n,p) = \Gamma(p).$$

On the other hand, by integrating by parts we get

$$\begin{aligned} f(n,p) &= \int_0^n \left(1-\frac{x}{n}\right)^n x^{p-1}dx \\ &= \left[\left(1-\frac{x}{n}\right)^n \frac{x^p}{p}\right]_0^n + \frac{1}{p}\int_0^n \left(1-\frac{x}{n}\right)^{n-1} x^p dx \\ &= \frac{1}{p}\int_0^n \left(1-\frac{x}{n}\right)^{n-1} x^p dx. \end{aligned}$$

Again, integrating by parts we get

$$\begin{aligned} f(n,p) &= \int_0^n \left(1-\frac{x}{n}\right)^n x^{p-1}dx \\ &= \frac{1}{p}\left[\left(1-\frac{x}{n}\right)^n \frac{x^{p+1}}{p+1}\right]_0^n + \frac{(n-1)}{np(p+1)}\int_0^n \left(1-\frac{x}{n}\right)^{n-2} x^{p+1}dx \\ &= \frac{n(n-1)}{n^2p(p+1)}\int_0^n \left(1-\frac{x}{n}\right)^{n-2} x^{p+1}dx. \end{aligned}$$

After integration by parts n times, we get

$$\begin{aligned} f(n,p) &= \frac{n(n-1)...[n-(n-1)]}{n^n p(p+1)(p+2)...[p+(n+1)]}\int_0^n \left(1-\frac{x}{n}\right)^{n-n} x^{p+(n-1)}dx \\ &= \frac{n!}{n^n p(p+1)(p+2)...[p+(n+1)]}\left[\frac{x^{n+p}}{n+p}\right]_0^n \\ &= \frac{n!n^p}{p(p+1)(p+2)...(p+n)}. \end{aligned}$$

Therefore

$$\Gamma(p) = \lim_{n\to+\infty} \frac{n!n^p}{p(p+1)(p+2)...(p+n)}.$$

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