1.2 Beta function

Definition 1.2.1 *The Beta function is a type of Euler integral defined by*

$$B(p,q)=\int_0^1 t^{p-1}(1-t)^{p+1}dt, p,q\in\mathbb{C}, \mathrm{Re}(p)>0, \mathrm{Re}(q)>0.$$

Proposition 1.2.1 *For all* $p,q\in\mathbb{C}$, *with* $\mathrm{Re}(p)>0, \mathrm{Re}(q)>0$, *we have*

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Proof. Let $D=(0,+\infty)\times(0,+\infty)$, we have

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \left(\int_0^{+\infty} x^{p-1}e^{-x}dx\right)\left(\int_0^{+\infty} y^{q-1}e^{-y}dy\right)\\ &= \int\int_D x^{p-1}y^{q-1}e^{-(x+y)}dxdy.\end{aligned}$$

Using a coordinate change, consider the new coordinates

$$\begin{cases} u=x+y \\ v=\frac{x}{x+y}\end{cases} \Longrightarrow \begin{cases} x=uv \\ y=u(1-v)\end{cases},$$

and

$$\frac{\partial(x,y)}{\partial(x,y)}=\begin{vmatrix} v & u \\ 1-v & -u\end{vmatrix}=-uv-u(1-v)=-u.$$

Just as the domain D' corresponding to D in the coordinates u,v is

$$D'=\{(u,v): u\geq 0, 0\leq v\leq 1\}.$$

So

$$\begin{aligned}\int\int_D x^{p-1}y^{q-1}e^{-(x+y)}dxdy &= \int\int_{D'}(uv)^{p-1}(u(1-v))^{q-1}e^{-u}\left|-u\right|dudv\\ &= \int\int_{D'} u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv\\ &= \int_0^{+\infty}\int_0^1 u^{p+q-1}v^{p-1}(1-v)^{q-1}e^{-u}dudv\\ &= \left(\int_0^{+\infty}u^{p+q-1}e^{-u}du\right)\left(\int_0^1 v^{p-1}(1-v)^{q-1}dv\right)\\ &= \Gamma(p+q)B(p,q).\end{aligned}$$

Therefore

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

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