1.3 Mittag-Leffler function

Definition 1.3.1 *The Mittag-Leffler function is defined by*

$$E_\alpha(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \alpha > 0,$$

and the generalized Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0.$$

Exemple 1.3.1

$$\begin{aligned} 1)\ E_{1,1}(x) &= E_1(x) \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+1)} \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{k!} \\ &= e^x. \end{aligned}$$

$$\begin{aligned} 2)\ E_{1,2}(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+2)} \\ &= \frac{1}{x} \sum_{k=0}^{+\infty} \frac{x^{k+1}}{(k+1)!} \\ &= \frac{1}{x}(e^x - 1). \end{aligned}$$

$$\begin{aligned} 3)\ E_{2,1}(x) &= E_2(x) \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(2k+1)} \\ &= \sum_{k=0}^{+\infty} \frac{x^k}{(2k)!} \\ &= \cosh \sqrt{x}. \end{aligned}$$

$$\begin{aligned} 4)\ E_{1,3}(x) &= \sum_{k=0}^{+\infty} \frac{x^k}{\Gamma(k+3)} \\ &= \frac{1}{x^2} \sum_{k=0}^{+\infty} \frac{x^{k+2}}{(k+2)!} \\ &= \frac{1}{x^2}(e^x - 1 - x). \end{aligned}$$

Proposition 1.3.1 *For $\alpha, \beta > 0, \lambda \in \mathbb{R}$ we have*

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right](s) = \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, s > 0, \left|\frac{\lambda}{s^{\alpha}}\right| < 1,$$

where $\mathcal{L}[.](s)$ is the Laplace transform [8].

Proof. Taking the Laplace transform of the function $t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})$, we obtain

$$\begin{aligned}\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right](s) &= \int_0^{\infty} t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})e^{-st}dt \\ &= \int_0^{\infty} t^{\beta-1}\sum_{k=0}^{\infty}\frac{(\lambda t^{\alpha})^k}{\Gamma(k\alpha+\beta)}e^{-st}dt \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)}\int_0^{\infty} t^{\alpha k+\beta-1}e^{-st}dt.\end{aligned}$$

Now, by integration by parts, we have

$$\int_0^{\infty} t^{\alpha k+\beta-1}e^{-st}dt = \frac{1}{s^{\alpha k+\beta}}\Gamma(k\alpha+\beta).$$

Therefore, we have

$$\begin{aligned}\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})\right](s) &= \sum_{k=0}^{\infty}\frac{\lambda^k}{\Gamma(k\alpha+\beta)}\frac{1}{s^{\alpha k+\beta}}\Gamma(k\alpha+\beta) \\ &= \sum_{k=0}^{\infty}\frac{\lambda^k}{s^{\alpha k+\beta}} \\ &= \frac{1}{s^{\beta}}\sum_{k=0}^{\infty}\left(\frac{\lambda}{s^{\alpha}}\right)^k \\ &= \frac{1}{s^{\beta}}\frac{1}{1-\frac{\lambda}{s^{\alpha}}}, \left|\frac{\lambda}{s^{\alpha}}\right| < 1 \\ &= \frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda}, \left|\frac{\lambda}{s^{\alpha}}\right| < 1.\end{aligned}$$

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