Chapter 2

Elements of fractional calculus

In this chapter, we present the elementary definitions of fractional integrals and fractional derivatives in the sense of Riemann-Liouville, Caputo and Grünwald-Letnikov. In addition its properties

2.1 Riemann-Liouville fractional integral

Functions defined on $[a, b]$.

Let $f : [a.b] \to \mathbb{R}$ a function defined on $[a, b]$.

Let's denote by $\left(I_{a^+}^1 f\right)$ the primitive of f which vanishes at a

$$\forall t \in [a, b] : \left(I_{a^+}^1 f\right)(t) = \int_a^t f(\tau) d\tau.$$

The iteration of f makes it possible to obtain the second primitive of f which vanishes at a and whose derivative vanishes at a. Furthermore, according to Fubini's theorem, we have

$$\begin{aligned}
\left(I_{a^+}^1 f\right)^2 (t) &= \left(\left(I_{a^+}^1 f\right) \circ \left(I_{a^+}^1 f\right)\right)(t) \\
&= \int_a^t \left(\int_a^u f(\tau) d\tau\right) du \\
&= \int_a^t (t - \tau) f(\tau) d\tau.
\end{aligned}$$

Let $n \in \mathbb{N}^*$, By denoting $\left(I_{a^+}^1 f\right)^n$ the n^{th} iteration of $\left(I_{a^+}^1 f\right)$, a direct recurrence shows that

$$\left(I_{a^+}^1 f\right)^n(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau.$$

If we note $g = \left(I_{a^+}^1 f\right)^n$, g is therefore the unique function verifying

$$\forall 0 \leq k \leq n-1 : g^{(k)}(a) = 0, g^{(n)} = f.$$

The equality $g^{(n)} = f$ justifies the following definition.

Definition 2.1.1 *Let $n \in \mathbb{N}^*$. The left integral of order n of f, which we note $\left(I_{a^+}^n f\right)$ is defined by*

$$\left(I_{a^+}^n f\right)(t) = \frac{1}{(n-1)!}\int_a^t (t-\tau)^{n-1} f(\tau)d\tau.$$

Thanks to the gamma function that we defined previously.

This is the property $\Gamma(n+1) = n!$, which allows us to generalize definition 2.1.1 as follows

Definition 2.1.2 *The left Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by*

$$\forall t \in [a,b] : \left(I_{a^+}^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

In the same way we define the Riemann-Liouville fractional integral on the right of order $\alpha > 0$ of f by

$$\forall t \in [a,b] : \left(I_{b^-}^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau.$$

Functions defined on $\mathbb{R}^+$ and $\mathbb{R}$.

It is natural to extend definition 2.1.2 to the $\mathbb{R}^+$ and $\mathbb{R}$ axes. Let us denote these operators

$$\forall t \in \mathbb{R}^+ : \left(I_{0^+}^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

$$\forall t \in \mathbb{R} : \left(I_{+}^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^t (t-\tau)^{\alpha-1} f(\tau)d\tau.$$

Proposition 2.1.1 *For $\alpha > 0, \beta > 0,$ we have*

$$1)\ I^{\alpha}_{a^+}(t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}.$$

$$2)\ I^{\alpha}_{b^-}(b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-t)^{\alpha+\beta-1}.$$

Proof. 1)

$$I^{\alpha}_{a^+}(t-a)^{\beta-1} = \int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\beta-1}d\tau.$$

Let's put

$$\tau - a = s(t-a) \Longrightarrow d\tau = (t-a)ds.$$

So we have

$$\begin{aligned}
I^{\alpha}_{a^+}(t-a)^{\beta-1} &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}(\tau-a)^{\beta-1}d\tau \\
&= \frac{1}{\Gamma(\alpha)}\int_0^1 ((t-a)-s(t-a))^{\alpha-1}(s(t-a))^{\beta-1}(t-a)ds \\
&= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta-1}\int_0^1 s^{\beta-1}(1-s)^{\alpha-1}ds \\
&= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta-1}B(\alpha,\beta) \\
&= \frac{1}{\Gamma(\alpha)}(t-a)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
&= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}.
\end{aligned}$$

2) Same idea (change of variable is $b-\tau = s(b-t)$). ■

Remark 2.1.1 *If $a = 0$ we have*

$$I^{\alpha}_{a^+}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}.$$

In particular according to this relation, the Riemann-Liouville fractional integral of a constant $C \in \mathbb{R}$ is given by

$$I^{\alpha}_{a^+}C = \frac{C}{\Gamma(\alpha+1)}t^{\alpha}.$$

Theorem 2.1.1 *If $f \in L^1([a,b])$, then $I^{\alpha}_{a^+}f$ exists for all $\alpha > 0$ and $I^{\alpha}_{a^+}f \in L^1([a,b])$.*

Proposition 2.1.2 *Let $\alpha > 0, \beta > 0$ and $f \in L^1([a,b])$. Then*

$$\left(I_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) = \left(I_{a^+}^{\beta} I_{a^+}^{\alpha} f\right)(t) = \left(I_{a^+}^{\alpha+\beta} f\right)(t).$$

Proof.

$$\begin{aligned}
\left(I_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \left(\frac{1}{\Gamma(\beta)} \int_a^{\tau} (\tau - s)^{\beta-1} f(s) ds\right) d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^{\tau} (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} f(s) ds d\tau.
\end{aligned}$$

Changing the integration order, we have

$$\begin{aligned}
\left(I_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_a^{\tau} (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} f(s) ds d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s) \left(\int_s^t (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} d\tau\right) ds
\end{aligned}$$

Let's put

$$\tau - s = u(t-s) \Longrightarrow d\tau = (t-s)du.$$

■

So we have

$$\begin{aligned}
\left(I_{a^+}^{\alpha} I_{a^+}^{\beta} f\right)(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s) \left(\int_s^t (t-\tau)^{\alpha-1} (\tau-s)^{\beta-1} d\tau\right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s) \left(\int_s^t ((t-s) - u(t-s))^{\alpha-1} (u(t-s))^{\beta-1} (t-s) du\right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s)\, (t-s)^{\alpha+\beta-1} \left(\int_0^1 (1-u)^{\alpha-1} u^{\beta-1} du\right) ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t f(s)\, (t-s)^{\alpha+\beta-1}\, ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1}\, f(s) ds \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-s)^{\alpha+\beta-1}\, f(s) ds \\
&= \left(I_{a^+}^{\alpha+\beta} f\right)(t).
\end{aligned}$$

2.2 Fractional derivatives

There are several definitions of fractional derivatives, we present in this part the definitions of Riemann-Liouville, Caputo as well as Grünwald-Letnikov which are the most used.