2.2.1 Fractional derivatives of Riemann-Liouville

If $\alpha > 0$, we note $[\alpha]$ the integer part of α, $[\alpha]$ is the unique integer verifying $[\alpha] \leq \alpha \leq [\alpha]+1$.

Let $f : [a, b] \to \mathbb{R}$. Inspired by the classical relation

$$\frac{d}{dt} = \frac{d^2}{dt^2} \circ I^1_{a^+},$$

we can define a fractional derivative of order $0 < \alpha \leq 1$ by

$$\frac{d^\alpha}{dt^\alpha} = \frac{d}{dt} \circ I^{1-\alpha}_{a^+}.$$

More generally, if $\alpha > 0$ and $n = [\alpha] + 1$, we can put

$$\frac{d^\alpha}{dt^\alpha} = \frac{d^n}{dt^n} \circ I^{n-\alpha}_{a^+}.$$

We obtain exactly the Riemann Liouville fractional derivative on the left.

Definition 2.2.1 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The left Riemann-Liouville fractional derivative of order α of f is defined by*

$$\forall t \in [a, b] : D^\alpha_{a^+} f(t) = \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{a^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau.$$

Furthermore, we saw that definition 2.2.1 of right integral was associated with $-d/dt$. The preceding reasoning therefore leads to the following definition

Definition 2.2.2 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The right Riemann-Liouville fractional derivative of order α of f is defined by*

$$\forall t \in [a, b] : D^\alpha_{b^-} f(t) = \left(-\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{b^-} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^b (\tau - t)^{n-\alpha-1} f(\tau) d\tau.$$

If now $f : \mathbb{R} \to \mathbb{R}$, the previous definitions generalize directly and are called Liouville fractional derivatives.

Definition 2.2.3 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The left Riemann-Liouville fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} : D^\alpha_{+} f(t) = \left(\frac{d}{dt}\right)^n \circ I^{n-\alpha}_{+} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{-\infty}^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau.$$

Furthermore, we saw that definition 2.2.3 of right integral was associated with $-d/dt$. The preceding reasoning therefore leads to the following definition

Definition 2.2.4 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The right Riemann-Liouville fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} : D_{-}^{\alpha} f(t) = \left(-\frac{d}{dt}\right)^{n} \circ I_{-}^{n-\alpha} f(t) = \frac{(-1)^{n}}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{t}^{+\infty} (\tau - t)^{n-\alpha-1} f(\tau) d\tau.$$

Remark 2.2.1 *1) For $\alpha = 0$ and $n = 1$, we have*

$$D_{a^{+}}^{0} f(t) = \frac{d}{dt} \circ I_{a^{+}}^{1} f(t) = f(t).$$

2) All these derivatives coincide with the usual derivatives for integer orders, $\forall n \in \mathbb{N}^{*}$ we have

$$D_{a^{+}}^{n} f(t) = D_{+}^{n} f(t) = \frac{d^{n}}{dt^{n}} f(t).$$

$$D_{b^{-}}^{n} f(t) = D_{-}^{n} f(t) = (-1)^{n} \frac{d^{n}}{dt^{n}} f(t).$$

Proposition 2.2.1 *For $\alpha > 0, \beta > 0$, we have*

$$1)\ D_{a^{+}}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}.$$

$$2)\ D_{b^{-}}^{\alpha} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}.$$

Proof. 1) We put $f(t) = (t-a)^{\beta-1}$, according to definition 2.2.1 and proposition 2.1.1 we have

$$D_{a^{+}}^{\alpha} f(t) = \left(\frac{d}{dt}\right)^{n} \circ I_{a^{+}}^{n-\alpha} f(t) = \left(\frac{d}{dt}\right)^{n} \left(\frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} (t-a)^{n-\alpha+\beta-1}\right),$$

and

$$\begin{aligned} \left(\frac{d}{dt}\right)^{n} (t-a)^{n-\alpha+\beta-1} &= (n-\alpha+\beta-1)(n-\alpha+\beta-2) \ldots (n-\alpha+\beta-1-(n-1))(t-a)^{-\alpha+\beta-1} \\ &= (n-\alpha+\beta-1)(n-\alpha+\beta-2) \ldots (\beta-\alpha)(t-a)^{\beta-\alpha-1}, \end{aligned}$$

and on the other hand

$$\begin{aligned} \Gamma(n-\alpha+\beta) &= (n-\alpha+\beta-1)\Gamma(n-\alpha+\beta-1) \\ &= (n-\alpha+\beta-1)(n-\alpha+\beta-2)\Gamma(n-\alpha+\beta-2) \\ &= (n-\alpha+\beta-1)(n-\alpha+\beta-2) \ldots (\beta-\alpha)\Gamma(\beta-\alpha). \end{aligned}$$

So

$$\begin{aligned} D^{\alpha}_{a^+} f(t) &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)} (n-\alpha+\beta-1)(n-\alpha+\beta-2)...(\beta-\alpha)(t-a)^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta)(n-\alpha+\beta-1)(n-\alpha+\beta-2)...(\beta-\alpha)}{(n-\alpha+\beta-1)(n-\alpha+\beta-2)...(\beta-\alpha)\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}. \end{aligned}$$

2) In the same way. ■

Remark 2.2.2 *For $\lambda = \beta - 1$ and $a = 0$ we have*

$$\begin{aligned} D^{\alpha}_{0^+} t^{\lambda} &= \frac{\Gamma(\lambda+1)}{\Gamma(n-\alpha+\lambda+1)} (n-\alpha+\lambda)(n-\alpha+\lambda-1)...(\lambda+1-\alpha)t^{\lambda-\alpha} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(n-\alpha+\lambda+1)} (n-(\alpha-\lambda))(n-1-(\alpha-\lambda))...(1-(\alpha-\lambda))t^{\lambda-\alpha} \\ &= \begin{cases} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha}, & si\ \alpha-\lambda \notin \{1,2,...,n\} \\ 0, & si\ \alpha-\lambda \in \{1,2,...,n\} \end{cases}, \quad \lambda > -1. \end{aligned}$$

If $\alpha - \lambda \in \{1,2,...,n\} \Longrightarrow \alpha - \lambda = m \Longrightarrow \lambda = \alpha - m, m \in \{1,2,...,n\}$ i.e.

$$D^{\alpha}_{0^+} t^{\alpha-m} = 0, m \in \{1,2,...,n\}.$$

Remark 2.2.3 *If $\beta = 0$ and $a = 0$, the Riemann-Liouville fractional derivative of a constant $C \in \mathbb{R}$ is non-zero and its value is*

$$D^{\alpha}_{a^+} C = \frac{C}{\Gamma(1-\alpha)} t^{-\alpha}.$$

2.2.2 Fractional derivatives of Caputo

This definition based on the inversion of the compositions in the formula of definition 2.2.1 also seems reasonable to define a fractional derivative called Caputo fractional derivative.

Definition 2.2.5 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The left Caputo fractional derivative of order α of f is defined by*

$$\forall t \in [a,b] :^{C} D^{\alpha}_{a^+} f(t) = I^{n-\alpha}_{a^+} \circ \left(\frac{d}{dt}\right)^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$