So

$$\begin{aligned} D^{\alpha}_{a^+}f(t) &= \frac{\Gamma(\beta)}{\Gamma(n-\alpha+\beta)}\,(n-\alpha+\beta-1)\,(n-\alpha+\beta-2)\,...(\beta-\alpha)(t-a)^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta)\,(n-\alpha+\beta-1)\,(n-\alpha+\beta-2)\,...(\beta-\alpha)}{(n-\alpha+\beta-1)(n-\alpha+\beta-2)...(\beta-\alpha)\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1} \\ &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}. \end{aligned}$$

2) In the same way. ■

Remark 2.2.2 *For $\lambda=\beta-1$ and $a=0$ we have*

$$\begin{aligned} D^{\alpha}_{0^+}t^{\lambda} &= \frac{\Gamma(\lambda+1)}{\Gamma(n-\alpha+\lambda+1)}\,(n-\alpha+\lambda)\,(n-\alpha+\lambda-1)\,...(\lambda+1-\alpha)t^{\lambda-\alpha} \\ &= \frac{\Gamma(\lambda+1)}{\Gamma(n-\alpha+\lambda+1)}\,(n-(\alpha-\lambda))\,(n-1-(\alpha-\lambda))\,...(1-(\alpha-\lambda))t^{\lambda-\alpha} \\ &= \begin{cases} \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}, & si\ \alpha-\lambda\notin\{1,2,...,n\} \\ 0, & si\ \alpha-\lambda\in\{1,2,...,n\} \end{cases}, \ \lambda>-1. \end{aligned}$$

If $\alpha-\lambda\in\{1,2,...,n\}\Longrightarrow\alpha-\lambda=m\Longrightarrow\lambda=\alpha-m, m\in\{1,2,...,n\}$ *i.e.*

$$D^{\alpha}_{0^+}t^{\alpha-m}=0, m\in\{1,2,...,n\}.$$

Remark 2.2.3 *If $\beta=0$ and $a=0$, the Riemann-Liouville fractional derivative of a constant $C\in\mathbb{R}$ is non-zero and its value is*

$$D^{\alpha}_{a^+}C=\frac{C}{\Gamma(1-\alpha)}t^{-\alpha}.$$

2.2.2 Fractional derivatives of Caputo

This definition based on the inversion of the compositions in the formula of definition 2.2.1 also seems reasonable to define a fractional derivative called Caputo fractional derivative.

Definition 2.2.5 *Let $\alpha>0$ and $n=[\alpha]+1$. The left Caputo fractional derivative of order α of f is defined by*

$$\forall t\in[a,b]:{}^{C}D^{\alpha}_{a^+}f(t)=I^{n-\alpha}_{a^+}\circ\left(\frac{d}{dt}\right)^{n}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau.$$

Furthermore, we saw that definition 2.2.5 of right integral was associated with $-d/dt$. The preceding reasoning therefore leads to the following definition

Definition 2.2.6 *Let $\alpha > 0$ and $n = [\alpha] + 1$. The right Caputo fractional derivative of order α of f is defined by*

$$\forall t \in [a,b] : {}^{C}D_{b^-}^{\alpha} f(t) = I_{b^-}^{n-\alpha} \circ \left(-\frac{d}{dt}\right)^n f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} f^{(n)}(\tau) d\tau.$$

Remark 2.2.4 *On the other hand, such definitions do not fit correctly with the classic derivatives, $\forall n \in \mathbb{N}^*$ we have*

$${}^{C}D_{a^+}^{n} f(t) = f^{(n}(t) - f^{(n}(a).$$

$${}^{C}D_{b^-}^{n} f(t) = (-1)^n \left(f^{(n}(t) - f^{(n}(a) \right).$$

Fortunately, the following result shows that they approach the classical derivatives by lower limit.

Lemma 2.2.1 *Let $\alpha \in \mathbb{R}^+ - \mathbb{N}$ and $n = [\alpha] + 1$. If $f \in AC^n([a,b])$, so almost everywhere*

$$\lim_{\alpha \to n^-} {}^{C}D_{a^+}^{\alpha} f(t) = f^{(n}(t).$$

$$\lim_{\alpha \to n^-} {}^{C}D_{b^-}^{\alpha} f(t) = (-1)^n f^{(n}(t).$$

Proposition 2.2.2 *For $\alpha > 0, \beta > 0$, we have*

$$1)\ {}^{C}D_{a^+}^{\alpha} (t-a)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \beta > n.$$

$$2)\ {}^{C}D_{b^-}^{\alpha} (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (b-t)^{\beta-\alpha-1}, \beta > n.$$

Proof. 1) We put $f(t) = (t-a)^{\beta-1}$, according to definition 2.2.5 and proposition 2.1.1 we have

$${}^{C}D_{a^+}^{\alpha} f(t) = I_{a^+}^{n-\alpha} \circ \left(\frac{d}{dt}\right)^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n (\tau - a)^{\beta-1} d\tau,$$

and

$$\begin{aligned} \left(\frac{d}{dt}\right)^n (\tau-a)^{\beta-1} &= (\beta-1)(\beta-2)\ldots(\beta-1-(n-1))(\tau-a)^{\beta-1-n} \\ &= (\beta-1)(\beta-2)\ldots(\beta-n)(\tau-a)^{\beta-n-1}. \end{aligned}$$

Hence

$$^{C}D_{a^+}^{\alpha}f(t)=\frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}(\tau-a)^{\beta-n-1}d\tau.$$

Let's put

$$\begin{aligned}\tau-a &= s(t-a)\Longrightarrow d\tau=(t-a)ds.\\ \tau &= -a-s(t-a)\end{aligned}$$

So we have

$$\begin{aligned}
^{C}D_{a^+}^{\alpha}f(t) &= \frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}\int_a^t(t-\tau)^{n-\alpha-1}(\tau-a)^{\beta-n-1}d\tau\\
&= \frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}\int_0^1((t-a)-s(t-a))^{n-\alpha-1}(s(t-a))^{\beta-n-1}(t-a)ds\\
&= \frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}(t-a)^{\beta-\alpha-1}\int_0^1(1-s)^{n-\alpha-1}s^{\beta-n-1}ds\\
&= \frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}(t-a)^{\beta-\alpha-1}B(n-\alpha,\beta-n)\\
&= \frac{(\beta-1)(\beta-2)...(\beta-n)}{\Gamma(n-\alpha)}\frac{\Gamma(n-\alpha)\Gamma(\beta-n)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}\\
&= \frac{(\beta-1)(\beta-2)...(\beta-n)\Gamma(\beta-n)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}\\
&= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1}.
\end{aligned}$$

2) In the same way. ■

Remark 2.2.5 *For $\lambda=\beta-1$ and $a=0$ we have*

$$\begin{aligned}
^{C}D_{0^+}^{\alpha}t^{\lambda} &= \frac{\lambda(\lambda-1)(\lambda-2)...(\lambda-(n-1))\Gamma(\lambda-(n-1))}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}\\
&= \begin{cases}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}t^{\lambda-\alpha}, & si\ \lambda\notin\{1,2,...,n-1\}\\ 0, & si\ \lambda\in\{1,2,...,n-1\}\end{cases},\ \lambda>-1.
\end{aligned}$$

i.e.

$$^{C}D_{0^+}^{\alpha}t^{m}=0, m\in\{1,2,...,n\}.$$

Remark 2.2.6 *The use of the formula for Caputo fractional derivative of order $\alpha>0$ of a constant $C\in\mathbb{R}$, expresses that this derivative is zero, i.e.*

$$D_{a^+}^{\alpha}C=0.$$

Theorem 2.2.1 *Let $\alpha \geq 0$ and $n = [\alpha] + 1$.*

If $f : [a; b] \to \mathbb{R}$ and if f has $(n-1)$ derivatives at a and $D^{\alpha}_{a^+} f(t)$ exists. So

$$ {}^{C}D^{\alpha}_{a^+} f(t) = D^{\alpha}_{a^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right], $$

almost for everything $t \in [a; b]$.

Proof. We have by definition

$$ \begin{aligned} D^{\alpha}_{a^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] &= \left(\frac{d}{dt} \right)^n I^{n-\alpha}_{a^+} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \\ &= \left(\frac{d}{dt} \right)^n \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left[f(\tau) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\tau-a)^k \right] d\tau. \end{aligned} $$

Using integration by part

$$ \begin{aligned} g(\tau) &= f(\tau) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\tau-a)^k \to \frac{d}{d\tau} g(\tau) = \frac{d}{d\tau} \left[f(\tau) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\tau-a)^k \right], \\ \frac{d}{dt} h(t) &= \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \qquad\qquad \to h(t) = -\frac{(t-\tau)^{n-\alpha}}{\Gamma(n-\alpha+1)}, \end{aligned} $$

we obtain

$$ \begin{aligned} I^{n-\alpha}_{a^+} g(t) &= \int_a^t \frac{(t-\tau)^{n-\alpha-1}}{\Gamma(n-\alpha)} \left[f(\tau) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (\tau-a)^k \right] d\tau \\ &= \left[-\frac{(t-\tau)^{n-\alpha}}{\Gamma(n-\alpha+1)} g(\tau) \right]_a^t + \int_a^t \frac{(t-\tau)^{n-\alpha}}{\Gamma(n-\alpha+1)} \frac{d}{d\tau} g(\tau) d\tau \\ &= I^{n-\alpha+1}_{a^+} \frac{d}{dt} g(t). \end{aligned} $$

Likewise for $n-$times

$$ \begin{aligned} I^{n-\alpha}_{a^+} g(t) &= I^{n-\alpha+n}_{a^+} \frac{d^n}{dt^n} g(t) \\ &= I^{n}_{a^+} I^{n-\alpha}_{a^+} \frac{d^n}{dt^n} g(t) \\ &= I^{n}_{a^+} I^{n-\alpha}_{a^+} \frac{d^n}{dt^n} \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \\ &= I^{n}_{a^+} I^{n-\alpha}_{a^+} \frac{d^n}{dt^n} f(t), \end{aligned} $$

because

$$\frac{d^n}{dt^n}\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]=0.$$

So

$$\begin{aligned}
D^{\alpha}_{a^+}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right] &= \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\\
&= \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}g(t)\\
&= \left(\frac{d}{dt}\right)^n I^{n}_{a^+}I^{n-\alpha}_{a^+}\frac{d^n}{dt^n}f(t)\\
&= I^{n-\alpha}_{a^+}\circ\frac{d^n}{dt^n}f(t)\\
&= {}^{C}D^{\alpha}_{a^+}f(t).
\end{aligned}$$

■

Corollaire 2.2.1 *Let $\alpha \geq 0, n=[\alpha]+1$ and $D^{\alpha}_{a^+}f,{}^{C}D^{\alpha}_{a^+}f$ exist, we assume that $f^{(k)}(a)=0$ for $k=0,1,...,n$. So*

$$D^{\alpha}_{a^+}f(t)={}^{C}D^{\alpha}_{a^+}f(t).$$

2.2.3 Fractional derivatives of Grünwald-Letnikov

This definition is based on obtaining derivatives by finite differences [9].

Let $f:\mathbb{R}\to\mathbb{R}$, for $h>0$ we have

$$f'(t)=\lim_{h\to 0}\frac{1}{h}\left[f(t)-f(t-h)\right],$$

and the second derivative

$$\begin{aligned}
f''(t) &= \lim_{h\to 0}\frac{1}{h}\left[f'(t)-f'(t-h)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{h}\left(f(t)-f(t-h)\right)-\frac{1}{h}\left(f(t-h)-f(t-2h)\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h^2}\left[f(t)-2f(t-h)+f(t-2h)\right].
\end{aligned}$$

More generally, the n^{th} derivative of f is given by

$$f^{(n)}(t)=\lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k C^n_k f(t-kh), \tag{2.2.1}$$