because
$$\frac{d^n}{dt^n}\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]=0.$$

So
$$\begin{aligned}
D^{\alpha}_{a^+}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right] &= \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]\\
&= \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{a^+}g(t)\\
&= \left(\frac{d}{dt}\right)^n I^{n}_{a^+}I^{n-\alpha}_{a^+}\frac{d^n}{dt^n}f(t)\\
&= I^{n-\alpha}_{a^+}\circ\frac{d^n}{dt^n}f(t)\\
&= {}^{C}D^{\alpha}_{a^+}f(t).
\end{aligned}$$

■

Corollaire 2.2.1 *Let $\alpha\geq 0, n=[\alpha]+1$ and $D^{\alpha}_{a^+}f, {}^C D^{\alpha}_{a^+}f$ exist, we assume that $f^{(k)}(a)=0$ for $k=0,1,...,n$. So*
$$D^{\alpha}_{a^+}f(t)={}^C D^{\alpha}_{a^+}f(t).$$

2.2.3 Fractional derivatives of Grünwald-Letnikov

This definition is based on obtaining derivatives by finite differences [9].

Let $f:\mathbb{R}\to\mathbb{R}$, for $h>0$ we have
$$f'(t)=\lim_{h\to 0}\frac{1}{h}\left[f(t)-f(t-h)\right],$$

and the second derivative
$$\begin{aligned}
f''(t) &= \lim_{h\to 0}\frac{1}{h}\left[f'(t)-f'(t-h)\right]\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{1}{h}\left(f(t)-f(t-h)\right)-\frac{1}{h}\left(f(t-h)-f(t-2h)\right)\right]\\
&= \lim_{h\to 0}\frac{1}{h^2}\left[f(t)-2f(t-h)+f(t-2h)\right].
\end{aligned}$$

More generally, the n^{th} derivative of f is given by
$$f^{(n)}(t)=\lim_{h\to 0}\frac{1}{h^n}\sum_{k=0}^{n}(-1)^k C^n_k f(t-kh), \tag{2.2.1}$$

with

$$C_k^n = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-(k-1))}{k!}.$$

It is possible to extend C_k^n to $k > n$, by putting $C_k^n = 0$.

Formula (2.2.1) then becomes

$$f^{(\alpha)}(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t - kh).$$

Here again, we can generalize the term on the right thanks to the gamma function, we put for $\alpha \in \mathbb{R}^+ - \mathbb{N}$ and $k \in \mathbb{N}$

$$C_k^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

Let us note this time that $C_k^\alpha \neq 0$ even if $k > n$.

Definition 2.2.7 *Let $\alpha > 0$. The left Grünwald-Letnikov fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} :^{GL} D_+^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t - kh).$$

Let us also define its analogue on the right.

Definition 2.2.8 *Let $\alpha > 0$. The right Grünwald-Letnikov fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} :^{GL} D_-^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t + kh).$$

The Grünwald-Letnikov fractional derivative is of obvious numerical interest. If h is sufficiently small, the discrete evaluation of $\frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t - kh)$ makes it possible to approximate the fractional derivative of Liouville on $\mathbb{R}$.

2.2.4 Properties of fractional operators

One of the interests of fractional calculus is that it also generalizes certain properties of classical derivatives and integrals: the fractional derivative of the integral of the same order gives the identity, the derivative of a derivative gives back under certain conditions a