

Chapter 1 : Probability basics reminder

1. Techniques of counting

In this section we present some techniques for determining without direct enumeration the number of possible outcomes of a particular experiment or the number of elements in a particular set. Such techniques are sometimes referred to as combinatorial analysis. We begin with the following basic principle.

Factorial notation

The product of the positive integers from 1 to n inclusive occurs very often in mathematics and hence is denoted by the special symbol $n!$ (read "n factorial") :

$$n! = 1.2.3 \dots .(n-2)(n-3)n.$$

It is also convenient to define $0! = 1! = 1$.

1.1. Permutations

Permutations without repetitions :

An arrangement of a set of n objects in a given order is called a permutation of the objects (taken all at a time). An arrangement of any $r \leq n$ of these objects in a given order is called an r -permutation or a permutation of the n objects taken r at a time. The number of permutations of n objects taken r at a time will be denoted by $P(n, r)$

Théorème 1. *The number of permutations of r objects selected from a set of n distinct objects is*

$$P(n, r) = n(n-1)(n-2) \dots (n-r+1),$$

or, in factorial notation,

$$P(n, r) = \frac{n!}{(n-r)!}.$$

In the case where all the objects are taken at a time, then

$$P(n, n) = n(n-1)(n-2) \dots, 1 = n!.$$

Exemple 1. An electronic controlling mechanism requires 5 distinct, but interchangeable, memory chips. In how many ways can this mechanism be assembled :

- (a) by placing the 5 chips in the 5 positions within the controller ?
- (b) by placing 3 chips in the odd numbered positions within the controller ?

Solution :

(a) When all 5 chips must be placed, the answer is $5!$. Alternatively, in the permutation notation with $n = 5$ and $r = 5$, the first formula yields

$$P(5, 5) = 5! = 5.4.3.2.1 = 120$$

(b) For $n = 5$ chips placed in $r = 3$ positions, the permutation is

$$P(5, 3) = \frac{5!}{2!} = 5.4.3 = 60.$$

Exemple 2. You are going on a road trip with 4 friends in a car that fits 5 people. How many different ways can everyone sit if you have to drive the whole way ?

Solution :

You have to sit in the driver's seat. Hence, there are 4 options for the 1st passenger seat. Once that person is seated, there are 3 options for the next passenger seat. This goes on until there is one person left with 1 seat.

$$1.P(4, 4) = 1.4! = 24.$$

Permutations with repetitions :

Frequently we want to know the number of permutations of objects some of which are alike, as illustrated below. The general formula follows.

Théorème 2. *The number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_k are alike is*

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

Exemple 3. How many different signals, each consisting of 8 flags hung in a vertical line, can be formed from a set of 4 indistinguishable red flags, 3 indistinguishable white flags, and a blue flag?

Solution :

We seek the number of permutations of 8 objects of which 4 are alike (the red flags) and 3 are alike (the white flags). By the above theorem, there are $\frac{8!}{4!3!} = 280$ different signals.

1.2. Combinations

There are many problems in which we must find the number of ways in which r objects can be selected from a set of n objects, **but we do not care about the order** in which the selection is made and where these objects are taken at a time. Therefore, to find the number of ways in which r objects can be selected from a set of n distinct objects, also called the number of combinations of n objects taken r at a time and denoted by $C(n, r)$.

Théorème 3. *The number of ways in which r objects can be selected from a set of n distinct objects is*

$$C(n, r) = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!},$$

or, in factorial notation,

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Note that $C(n, n) = 1$ and $C(n, 0) = 1$ since there is only one way to choose a set of (all) n elements or no elements, and $C(n, 1) = n$ since there are n subsets of size 1.

Exemple 4. There are 20 hockey players on a pro NHL team, 2 of whom are goalies. How many sets of 5 skaters and 1 goalie can be on the ice at the same time?

Solution :

The question asks for how many on the ice, implying that order does not matter. This is combination problem with 2 combinations. You need to choose 1 goalie out of a possible of 2, and choose 5 skaters out of a possible 18.

$$C(2, 1)C(18, 5) = 2 \cdot \frac{18!}{5!13!}$$

Exemple 5. How many different ways could you score a 70% on a 10-question test, where each question is weighted equally and is either right or wrong?

Solution :

The order of the questions you got right does not matter, so this is a combination problem,

$$C(10, 7) = \frac{10!}{7!3!} = 120.$$

Ordered partitions

Théorème 4. *Let A contain n elements and let n_1, n_2, \dots, n_r be positive integers with $n_1 + n_2 + \dots + n_r = n$. Then there exist $\frac{n!}{n_1!n_2!\dots n_r!}$ different ordered partitions of A of the form (A_1, A_2, \dots, A_r) where A_1 contains n_1 elements, A_2 contains n_2 elements, ..., and A_r contains n_r elements.*

Example 6. In how many ways can 9 toys be divided between 4 children if the youngest child is to receive 3 toys and each of the other children 2 toys?

Solution :

We wish to find the number of ordered partitions of the 9 toys into 4 cells containing 3, 2, 2 and 2 toys respectively. By the above theorem, there are

$$C(9, 3).C(6, 2).C(4, 2).C(2, 2) = \frac{9!}{3!2!2!2!} = 7560$$

such ordered partitions.

2. Sample spaces, events, and their probabilities

Sample spaces

A **random experiment** is a mechanism that produces a definite outcome that cannot be predicted with certainty. The **sample space** S , is the set of all possible outcomes of a random experiment. Each outcome in a sample space is called an element or a member of the sample space.

For example, there are only two outcomes for tossing a coin, and the sample space is

$$S = \{Heads, Tails\} = \{H, T\}.$$

If we toss a coin three times, then the sample space is

$$S = \{HHH, HHT, HTH, THH, HTT, TTH, THT, TTT\}.$$

Example 7. Toss a coin until a head appears and then count the number of times the coin was tossed. The sample space of this experiment is $S = \{1, 2, 3, \dots, \infty\}$. Here ∞ refers to the case when a head never appears and so the coin is tossed an infinite number of times. This is an example of a sample space which is countably infinite.

Example 8. Consider rolling a fair die twice and observing the dots facing up on each roll. There are 36 possible outcomes in the sample space S , where

$$S = \{(1, 1), (1, 2), \dots, (1, 6), (2, 1), (2, 2), \dots, (2, 6), \dots, (6, 1), \dots, (6, 6)\}$$

Events

An event is a subset of a sample space S . An event A is said to occur on a particular trial of the experiment if the outcome observed is an element of the set A .

Example 9. 1-Construct a sample space for the experiment that consists of rolling a single die.

2-Find the events that correspond to the phrases A : "an even number is rolled" and B : "a number greater than two is rolled."

Solution :

1- The outcomes could be labelled according to the number of dots on the top face of the die. Then the sample space is the set $S = \{1, 2, 3, 4, 5, 6\}$.

2- The outcomes that corresponds to the event A is the set $\{2, 4, 6\}$, so we set $A = \{2, 4, 6\}$. Similarly the outcomes that corresponds to event B is the set $\{3, 4, 5, 6\}$, and we set $B = \{3, 4, 5, 6\}$.

2.1. Probability of an event

Probability is a real-valued function P that assigns to each event A in a sample space S a number called the probability of the event A , denoted by $P(A)$, such that the following three properties are satisfied :

A1 - For every event A , $0 \leq P \leq 1$,

A2 - $P(S) = 1$,

A3 - If A and B are disjoint events(i.e, $A \cap B = \phi$), then

$$P(A \cup B) = P(A) + P(B),$$

A4 - If A_1, A_2, A_3, \dots is a sequence of disjoint events, then

$$P(A_1 \cup A_2 \cup \dots) = P(A_1) + P(A_2) + \dots$$

Based on the above axioms, we can derive many other results which play important roles in applications :

- Théorème 5.**
- I- If ϕ is the empty set, then $P(\phi) = 0$.
 - II- If A is any event in S , then $P(A^c) = 1 - P(A)$, A^c is the complement of an event A .
 - III- If A and B are any two events, then $P(A - B) = P(A) - P(A \cap B)$.
 - IV- If $A \subset B$, then $P(A) \leq P(B)$.
 - V- If A and B are any events in S , then : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Démonstration.

- I- Since we have S and ϕ are disjoint events, then from A2 and A3 $P(S) = P(S \cup \phi) = P(S) + P(\phi) = 1$. Then $P(\phi) = 0$.
- II- Since $S = A \cup A^c$, and A and A^c are disjoint, then from A3 it results that : $P(S) = P(A) + P(A^c) = 1$. Hence, $P(A^c) = 1 - P(A)$.
- III- The event A can be decomposed into the mutually disjoint events $A - B$ and $A \cap B$; that is, $A = (A - B) \cup (A \cap B)$. Thus by A3, $P(A) = P(A - B) + P(A \cap B)$, from which our result follows.
- IV- Since $B = A \cup (B \cap A^c)$, then from A3, $P(B) = P(A) + P(B \cap A^c) \geq P(A)$.
- V- Note that the event $A \cup B$ can be decomposed into disjoint events $A - B$ and B ; that is, $A \cup B = (A - B) \cup B$. Thus by A3, $P(A \cup B) = P(A - B) + P(B) = P(A) - P(A \cap B) + P(B)$, which is the desired result. \square

2.2. Independence of events

Définition 1. Two events A and B are independent if and only if their joint probability equals the product of their probabilities

$$P(A \cap B) = P(A).P(B)$$

$A \cap B \neq \phi$ indicates that two independent events A and B have common elements in their sample space so that they are not mutually exclusive (disjoint).

Exemple 10. The probability that Tom forgets his PE kit is 0,3. The probability that Noah forgets his PE kit is 0,1. The events are independent. Calculate the probability that both Tom and Noah forget their PE kits on the same day.

Solution

The first event is that Tom forgets his PE kit. The probability that Tom forgets his PE kit is 0,3. The second event is that Noah forgets his PE kit. The probability that Noah forgets his PE kit is 0,1. The probability that Tom and Noah both forget their PE kits is $0,3 \times 0,1 = 0,03$.

Exemple 11. Rachel tosses three fair coins. Find the probability that all three coins land on tails.

Solution

The outcome for each coin is not affected by the other coins, therefore the events are independent. Since the coins are fair, for each coin the probability that it lands on tails is 0,5. Hence, the probability of getting three tails is $0,5^3 = 0,125$.

3. Conditional probability and Bayes' theorem

Conditional probability

The conditional probability of an event B is the probability that the event will occur given the knowledge that an event A has already occurred i.e., $P(A) > 0$. This probability is written $P(B|A)$: the probability of B given A , is defined as follows :

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

- In the case where events A and B are independent, the conditional probability of event B given event A is simply the probability of event B , that is $P(B)$.

- If events A and B are dependent, then the probability of the intersection of A and B (the probability that both events occur) is defined by $P(A \cap B) = P(A)P(B|A)$.

Example 12. You toss a fair coin three times. Given that you have observed at least one head, what is the probability that you observe at least two heads?

Solution

Let A_1 be the event that you observe at least one head, and A_2 be the event that you observe at least two heads. Then

$$A_1 = S - \{TTT\}, \text{ and } P(A_1) = \frac{7}{8};$$

$$A_2 = \{HHT, HTH, THH, HHH\}, \text{ and } P(A_2) = \frac{4}{8}.$$

Thus, we can write

$$\begin{aligned} P(A_2|A_1) &= \frac{P(A_2 \cap A_1)}{P(A_1)} \\ &= \frac{P(A_2)}{P(A_1)} \\ &= \frac{4}{8} \cdot \frac{8}{7} = \frac{4}{7} \end{aligned}$$

3.1. Bayes' Rule

Now we are ready to state one of the most useful results in conditional probability : Bayes' rule. Suppose that we know $P(A|B)$, but we are interested in the probability $P(B|A)$. Using the definition of conditional probability, we have

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A).$$

Dividing by $P(A)$, we obtain

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

which is the famous Bayes' rule. Often, in order to find $P(A)$ in Bayes' formula we need to use the law of total probability, so sometimes Bayes' rule is stated as

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)},$$

where B_1, B_2, \dots, B_n form a partition of the sample space.

Définition 2. (Bayes' Rule).

- For any two events A and B , where $P(A) \neq 0$, we have

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}.$$

- If B_1, B_2, B_3, \dots form a partition of the sample space S , and A is any event with $P(A) \neq 0$, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^n P(A|B_i)P(B_i)}.$$

Example 13. Three machines A, B and C produce respectively 50%, 30% and 20% of the total number of items of a factory. The percentages of defective output of these machines are 3%, 4% and 5%. If an item is selected at random, find the probability that the item is defective.

Solution

Let D be the event that an item is defective. Then using Bayes' formula and as illustrated in Figure 1 we get,

$$\begin{aligned} P(D) &= P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C) \\ &= (0.5)(0.03) + (0.3)(0.04) + (0.2)(0.05) = 0.037 \end{aligned}$$

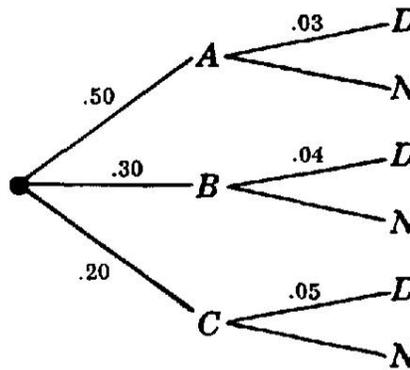


FIGURE 1: Diagram of Example 13.

Example 14. Consider the factory in the preceding example. Suppose an item is selected at random and is found to be defective. Find the probability that the item was produced by machine A; that is, find $P(A|D)$.

Solution

By Bayes' theorem,

$$\begin{aligned} P(A|D) &= \frac{P(A)P(D|A)}{P(A)P(D|A) + P(B)P(D|B) + P(C)P(D|C)} \\ &= \frac{(0.5)(0.03)}{(0.5)(0.03) + (0.3)(0.04) + (0.2)(0.05)} = \frac{15}{37} \end{aligned}$$