with

$$C_k^n = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)...(n-(k-1))}{k!}.$$

It is possible to extend C_k^n to $k > n$, by putting $C_k^n = 0$.

Formula (2.2.1) then becomes

$$f^{(\alpha)}(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t-kh).$$

Here again, we can generalize the term on the right thanks to the gamma function, we put for $\alpha \in \mathbb{R}^+ - \mathbb{N}$ and $k \in \mathbb{N}$

$$C_k^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}.$$

Let us note this time that $C_k^\alpha \neq 0$ even if $k > n$.

Definition 2.2.7 *Let $\alpha > 0$. The left Grünwald-Letnikov fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} :^{GL} D_+^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t-kh).$$

Let us also define its analogue on the right.

Definition 2.2.8 *Let $\alpha > 0$. The right Grünwald-Letnikov fractional derivative of order α of f is defined by*

$$\forall t \in \mathbb{R} :^{GL} D_-^\alpha f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t+kh).$$

The Grünwald-Letnikov fractional derivative is of obvious numerical interest. If h is sufficiently small, the discrete evaluation of $\frac{1}{h^\alpha} \sum_{k=0}^{+\infty} (-1)^k C_k^\alpha f(t-kh)$ makes it possible to approximate the fractional derivative of Liouville on $\mathbb{R}$.

2.2.4 Properties of fractional operators

One of the interests of fractional calculus is that it also generalizes certain properties of classical derivatives and integrals: the fractional derivative of the integral of the same order gives the identity, the derivative of a derivative gives back under certain conditions a derivative, the integration by parts remains valid and fractional operators combine very well with Fourier and Laplace transforms. This last property is omnipresent in many fields of applications presented in the previous section.

Linearity

Fractional integration and differentiation are linear operators

$$I^{\alpha}\left(\lambda f(t)+\mu g(t)\right)=\lambda I^{\alpha}\left(f(t)\right)+\mu I^{\alpha}\left(g(t)\right), \lambda, \mu \in \mathbb{R},$$

$$D^{\alpha}\left(\lambda f(t)+\mu g(t)\right)=\lambda D^{\alpha}\left(f(t)\right)+\mu D^{\alpha}\left(g(t)\right), \lambda, \mu \in \mathbb{R},$$

for any derivation approach.

Compositions between operators

Proposition 2.2.3 *Let* $\alpha>0, \beta>0$ *and* $n=[\alpha]+1$ *we have the following properties*

1) If $f(t) \in L^{p}([a, b]), 1 \leq p \leq \infty,$ *then*

$$\left(D_{a^{+}}^{\alpha} I_{a^{+}}^{\alpha} f\right)(t)=f(t) \text{ and } \left(D_{b^{-}}^{\alpha} I_{b^{-}}^{\alpha} f\right)(t)=f(t).$$

2) If $\alpha>\beta$ *and* $f(t) \in L^{p}([a, b]), 1 \leq p \leq \infty,$ *then*

$$\left(D_{a^{+}}^{\beta} I_{a^{+}}^{\alpha} f\right)(t)=\left(I_{a^{+}}^{\alpha-\beta} f\right)(t) \text{ and } \left(D_{b^{-}}^{\beta} I_{b^{-}}^{\alpha} f\right)(t)=\left(I_{b^{-}}^{\alpha-\beta} f\right)(t).$$

3) If $f(t) \in C^{q}([a, b]), q=[\alpha+\beta]+1,$ *then*

$$\left(D_{a^{+}}^{\alpha} D_{a^{+}}^{\beta} f\right)(t)=\left(D_{a^{+}}^{\alpha+\beta} f\right)(t) \text{ and } \left(D_{b^{-}}^{\alpha} D_{b^{-}}^{\beta} f\right)(t)=\left(D_{b^{-}}^{\alpha+\beta} f\right)(t).$$

4) If $f(t) \in L^{1}([a, b])$ *and* $\left(I_{a^{+}}^{n-\alpha} f\right)(t) \in AC^{n}([a, b]),$ *then*

$$\left(I_{a^{+}}^{\alpha} D_{a^{+}}^{\alpha} f\right)(t)=f(t)-\sum_{k=0}^{n} \frac{\left(I_{a^{+}}^{n-\alpha} f\right)^{(n-k)}(a)}{\Gamma(\alpha-k+1)}(t-a)^{\alpha-k},$$

$$\left(I_{b^{-}}^{\alpha} D_{b^{-}}^{\alpha} f\right)(t)=f(t)-\sum_{k=0}^{n} \frac{(-1)^{n-k}\left(I_{b^{-}}^{n-\alpha} f\right)^{(n-k)}(b)}{\Gamma(\alpha-k+1)}(b-t)^{\alpha-k}.$$

In particular if $0<\alpha \leq 1$

$$\left(I_{a^{+}}^{\alpha} D_{a^{+}}^{\alpha} f\right)(t)=f(t),$$

$$\left(I_{b^{-}}^{\alpha} D_{b^{-}}^{\alpha} f\right)(t)=f(t).$$

Proposition 2.2.4 *Let $\alpha > 0, \beta > 0$ and $n = [\alpha] + 1$ we have the following properties*

1) If $f(t) \in C^q([a,b]), q = [\alpha + \beta] + 1,$ then

$$\left({}^C D^{\alpha}_{a+} \ {}^C D^{\beta}_{a+} f(t)\right)(t) = \left({}^C D^{\alpha+\beta}_{a+} f\right)(t) \text{ and } \left({}^C D^{\alpha}_{b-} \ {}^C D^{\beta}_{b-} f(t)\right)(t) = \left({}^C D^{\alpha+\beta}_{b-} f\right)(t).$$

2) If $f(t) \in C^n([a,b])$ or $f(t) \in AC^n([a,b]),$ then

$$\left(I^{\alpha}_{a+} \ {}^C D^{\alpha}_{a+} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k,$$

$$\left(I^{\alpha}_{b-} \ {}^C D^{\alpha}_{b-} f\right)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-t)^k.$$

In particular if $0 < \alpha \leq 1$

$$\left(I^{\alpha}_{a+} \ {}^C D^{\alpha}_{a+} f\right)(t) = f(t) - f(a),$$

$$\left(I^{\alpha}_{b-} \ {}^C D^{\alpha}_{b-} f\right)(t) = f(t) - f(b).$$

Integration by parts

The integration by parts formula is one of the properties extensible to fractional operators but again under certain restrictions. This is where operators inevitably appear on the right. In [12] a formula for integration by parts appears, but it requires several conditions. We prefer to give here a simplified version with explicit conditions that we found in [1].

$$\int_a^b f(t) D^{\alpha}_{a+} g(t) dt = \int_a^b D^{\alpha}_{b-} f(t) g(t) dt,$$

$$\int_a^b f(t) D^{\alpha}_{b-} g(t) dt = \int_a^b D^{\alpha}_{a+} f(t) g(t) dt.$$

Fourier transform of fractional derivatives

The Fourier transform of a function $f \to L^1(\mathbb{R})$ can be defined by [2]

$$\forall \omega \in \mathbb{R} : \mathcal{F}[f](\omega) = \int_{-\infty}^{+\infty} f(t) e^{-it\omega} dt.$$

Let $n \in \mathbb{N}$. If f as well as all its derivatives up to order n are integrable, then

$$\mathcal{F}\left[f^{(n)}\right](\omega) = (i\omega)^n \mathcal{F}[f](\omega).$$

This result generalizes to fractional operators defined on $\mathbb{R}$.

Lemma 2.2.2 *Let $0 < \alpha \leq 1$ and $f \in L^1(\mathbb{R})$. So*

$$\mathcal{F}\left[I_{\pm}^{\alpha} f\right](\omega) = (\pm i\omega)^{-\alpha}\, \mathcal{F}[f](\omega).$$

Corollaire 2.2.2 *Let $\alpha > 0$ and $n = [\alpha] + 1$. Let $f \in L^1(\mathbb{R})$ such as for everything $1 \leq k \leq n$, $D_{\pm}^{k+\alpha-n} f \in L^1(\mathbb{R})$. So*

$$\mathcal{F}\left[D_{\pm}^{\alpha} f\right](\omega) = (\pm i\omega)^{\alpha}\, \mathcal{F}[f](\omega).$$

Proof. According to Lemma 2.2.2

$$\mathcal{F}\left[I_{\pm}^{n-\alpha} f\right](\omega) = (\pm i\omega)^{\alpha-n}\, \mathcal{F}[f](\omega).$$

As for all $1 \leq k \leq n$, $\frac{d^k}{dt^k} I_{+}^{n-\alpha} f = D_{+}^{k+\alpha-n} f \in L^1(\mathbb{R})$, so

$$\begin{aligned}
\mathcal{F}\left[D_{+}^{\alpha} f\right](\omega) &= \mathcal{F}\left[\frac{d^n}{dt^n} I_{+}^{n-\alpha} f\right](\omega) \\
&= (i\omega)^n\, \mathcal{F}\left[I_{+}^{n-\alpha} f\right](\omega) \\
&= (i\omega)^n\, (i\omega)^{\alpha-n}\, \mathcal{F}[f](\omega) \\
&= (i\omega)^{\alpha}\, \mathcal{F}[f](\omega).
\end{aligned}$$

The same for everything $1 \leq k \leq n$, $\frac{d^k}{dt^k} I_{-}^{n-\alpha} f = (-1)^k D_{-}^{k+\alpha-n} f \in L^1(\mathbb{R})$, so

$$\begin{aligned}
\mathcal{F}\left[D_{-}^{\alpha} f\right](\omega) &= \mathcal{F}\left[(-1)^n \frac{d^n}{dt^n} I_{-}^{n-\alpha} f\right](\omega) \\
&= (-1)^n\, (i\omega)^n\, \mathcal{F}\left[I_{-}^{n-\alpha} f\right](\omega) \\
&= (-i\omega)^n\, (-i\omega)^{\alpha-n}\, \mathcal{F}[f](\omega) \\
&= (-i\omega)^{\alpha}\, \mathcal{F}[f](\omega).
\end{aligned}$$

■

Laplace transform of fractional derivatives

We say that the real function $f : \mathbb{R}^+ \to \mathbb{R}^N$ has sub-exponential increasing, if

$$\exists A > 0, \exists s_0 \in \mathbb{R}, \exists t_0 > 0, \forall t > t_0 : |f(t)| \leq e^{s_0 t}.$$

If $f \in L^1(\mathbb{R}^+)$, is sub-exponential increasing, recall that its Laplace transform is defined by [13]

$$\forall s > s_0,\ \mathcal{L}[f](s) = \int_0^{+\infty} f(t) e^{-st} dt.$$

For $n \in \mathbb{N}$, if $f \in C^n(\mathbb{R}^+)$ is sub-exponential increasing, then

$$\forall s > s_0,\ \mathcal{L}\left[f^{(n)}\right](s) = s^n \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0). \tag{2.2.2}$$

The extension to the fractional case is carried out this time with the fractional operators with supports reduced by 0.

Lemma 2.2.3 *Let $\alpha > 0$ and $f \in L^1(\mathbb{R})$ is sub-exponential increasing. Then*

$$\mathcal{L}[I_0^\alpha f](s) = s^{-\alpha} \mathcal{L}[f](s).$$

Proof. We write the Riemann-Liouville fractional integral $I_0^\alpha f$ as a convolution of two functions, i.e.

$$\begin{aligned} (I_0^\alpha f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t). \end{aligned}$$

So

$$\begin{aligned} \mathcal{L}[I_0^\alpha f](s) &= \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f\right](s) \\ &= \mathcal{L}\left[\frac{1}{\Gamma(\alpha)} t^{\alpha-1}\right](s) \mathcal{L}[f](s) \\ &= s^{-\alpha} \mathcal{L}[f](s). \end{aligned}$$

■

Proposition 2.2.5 *Let $\alpha > 0$ and $n = [\alpha]+1$ and $f \in C^n(\mathbb{R})$ is sub-exponential increasing. Then*

$$1)\ \forall s > s_0,\ \mathcal{L}[D_0^\alpha f](s) = s^\alpha \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^\alpha \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^k D^{\alpha-k-1} f(0).$$

$$2)\ \forall s > s_0,\ \mathcal{L}\left[{}^C D_0^\alpha f\right](s) = s^\alpha \mathcal{L}[f](s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0).$$

Proof. 1) We apply (2.2.2) to f, then we use Lemma 2.2.3

$$\begin{aligned}\mathcal{L}\left[D_0^\alpha f\right](s) &= \mathcal{L}\left[\frac{d^n}{dt^n} I_0^{n-\alpha} f\right](s)\\ &= s^n \mathcal{L}\left[I_0^{n-\alpha} f\right](s) - \sum_{k=0}^{n-1} s^{n-k-1} (I_0^{n-\alpha} f)^{(k)}(0)\\ &= s^n s^{\alpha-n} \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{n-k-1} (I_0^{n-\alpha} f)^{(k)}(0)\\ &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{n-k-1} (I_0^{n-\alpha} f)^{(k)}(0).\end{aligned}$$

We put

$$g(t) = (I_0^{n-\alpha} f)(t). \tag{2.2.3}$$

So, we have

$$\begin{aligned}\mathcal{L}\left[D_0^\alpha f\right](s) &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{n-k-1} (I_0^{n-\alpha} f)^{(k)}(0)\\ &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{n-k-1} g^{(k)}(0)\\ &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{k} g^{(n-k-1)}(0).\end{aligned}$$

From (2.2.3), we have

$$\begin{aligned}g^{(n-k-1)}(0) &= \frac{d^{n-k-1}}{dt^{n-k-1}} g(0)\\ &= D^{n-k-1} I^{n-\alpha} f(0)\\ &= D^{\alpha-k-1} f(0).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{L}\left[D_0^\alpha f\right](s) &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{k} g^{(n-k-1)}(0)\\ &= s^\alpha \mathcal{L}\left[f\right](s) - \sum_{k=0}^{n-1} s^{k} D^{\alpha-k-1} f(0).\end{aligned}$$

In the same way we apply Lemma 2.2.3 to $f^{(n)}$, then we use (2.2.2)

$$\begin{aligned}\mathcal{L}\left[{}^{C}D_{0}^{\alpha}f\right](s) &= \mathcal{L}\left[I_{0}^{n-\alpha}\frac{d^{n}}{dt^{n}}f\right](s) \\ &= s^{\alpha-n}\mathcal{L}\left[f^{(n)}\right](s) \\ &= s^{\alpha-n}\left[s^{n}\mathcal{L}\left[f\right](s)-\sum_{k=0}^{n-1}s^{n-k-1}f^{(k)}(0)\right] \\ &= s^{\alpha}\mathcal{L}\left[f\right](s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0).\end{aligned}$$

■

Remark 2.2.7 *Note the absence of generalization for the derivative of the product and the composition of two functions. These characteristics of the classical derivative actually transfer poorly to the fractional. Regardless of the definition used and even with restrictions on functions*

$$D^{\alpha}(fg) \neq D^{\alpha}(f)g + fD^{\alpha}(g),$$

$$D^{\alpha}\left(\frac{f}{g}\right) \neq \frac{D^{\alpha}(f)g - fD^{\alpha}(g)}{g^{2}},$$

$$D^{\alpha}(f\circ g) \neq D^{\alpha}(f)\,(g)\,.g'.$$