4.3 Boundary problem of fractional differential equation

In this part we will study the existence and uniqueness of the solution of a fractional differential equation limit problem in forms following form

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t,u(t)) \\ au(0) + bu(T) = c \end{cases}, \tag{4.3.1}$$

where $t \in [0,T], 0 < \alpha \leq 1,\ f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and a, b, c real constants with $a + b \neq 0$.

Definition 4.3.1 *We say that a function $u \in C^1([0,T],\mathbb{R})$, is solution of the problem (4.3.1) if u verify the equation ${}^{C}D^{\alpha}u(t) = f(t,u(t))$, on A and with the condition $au(0) + bu(T) = c$.*

Lemma 4.3.1 *Let $0 < \alpha \leq 1$ and let $h : [0,T] \longrightarrow \mathbb{R}$ is a continuous function. A function u is a solution to the boundary problem*

$$\begin{cases} {}^{C}D^{\alpha}u(t) = h(t), t \in [0,T], 0 < \alpha \leq 1 \\ au(0) + bu(T) = c \end{cases}, \tag{4.3.2}$$

if and only if it is the solution of the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds - \frac{1}{a+b} \left[\frac{b}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds - c \right]. \tag{4.3.3}$$

Proof. We apply the operator I^{α} to equation (4.3.2) we find

$$\begin{aligned} I^{\alpha}\, {}^{C}D^{\alpha}u(t) &= I^{\alpha}h(t) \\ &\Longrightarrow u(t) + c_0 = I^{\alpha}h(t) \\ &\Longrightarrow u(t) = I^{\alpha}h(t) - c_0. \end{aligned}$$

According to the boundary conditions we have

$$\begin{aligned} u(0) &= I^{\alpha}h(0) - c_0 = -c_0, \\ u(T) &= I^{\alpha}h(T) - c_0. \end{aligned}$$

So

$$\begin{aligned} au(0)+bu(T) &= -ac_0 + b\left[I^\alpha h(T) - c_0\right] = c \\ &\Longrightarrow ac_0 + bc_0 = bI^\alpha h(T) - c \\ &\Longrightarrow c_0 = \frac{1}{a+b}\left[bI^\alpha h(T) - c\right], a+b \neq 0. \end{aligned}$$

Then

$$\begin{aligned} u(t) &= I^\alpha h(t) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}h(s)ds - c\right]. \end{aligned}$$

Conversely we have

$$u(t) = I^\alpha h(t) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right].$$

We apply ${}^C D^\alpha$ to the integral equation (4.3.3)

$$\begin{aligned} {}^C D^\alpha u(t) &= {}^C D^\alpha I^\alpha h(t) -{}^C D^\alpha\left[\frac{1}{a+b}\left[bI^\alpha h(T) - c\right]\right] \\ &= h(t). \end{aligned}$$

All that remains is to verify that $au(0) + bu(T) = c$

$$\begin{cases} u(0) = I^\alpha h(0) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right] \\ u(T) = I^\alpha h(T) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right] \end{cases}.$$

So

$$\begin{aligned} au(0)+bu(T) &= a\left[I^\alpha h(0) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right] + b\left[I^\alpha h(T) - \frac{1}{a+b}\left[bI^\alpha h(T) - c\right]\right]\right] \\ &= -\frac{a}{a+b}\left[bI^\alpha h(T) - c\right] + bI^\alpha h(T) - \frac{b}{a+b}\left[bI^\alpha h(T) - c\right] \\ &= -\frac{a+b}{a+b}\left[bI^\alpha h(T) - c\right] + bI^\alpha h(T) \\ &= -bI^\alpha h(T) + c + bI^\alpha h(T) \\ &= c \end{aligned}$$

Then there is a solution to the problem (4.3.2). ∎

Theorem 4.3.1 *Let $0 < \alpha \leq 1$ and $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ and verifies the following Lipschitz condition*

$$|f(t,u) - f(t,v)| \leq k\,|u - v|\,, \forall t \in [0,T] \text{ and } u, v \in \mathbb{R}.$$

If

$$\frac{kT^{\alpha}\left(1 + \frac{|b|}{|a+b|}\right)}{\Gamma(\alpha+1)} < 1,$$

then the problem (4.3.1) admits a unique solution on $[0,T]$.

Proof. We transform the problem (4.3.1) into a fixed point problem, Consider the operator

$$F : C([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R}),$$

defined by

$$Fu(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds - \frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f(s,u(s))ds - c\right]. \tag{4.3.4}$$

So the fixed points of F are the solutions of the problem (4.3.1) we have:

F is well defined, in fact: if $u \in C([0,T],\mathbb{R})$ then $Fu(t) \in C([0,T],\mathbb{R})$, so if showing F is contraction then F admits a point fixed in effect if $u_1, u_2 \in C([0,T],\mathbb{R})$, then $\forall t \in [0,T]$, we have

$$\begin{aligned}
|Fu_1(t) - Fu_2(t)| &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,|f(s,u_1(s)) - f(s,u_2(s))|\,ds \\
&\quad + \frac{|b|}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}\,|f(s,u_1(s)) - f(s,u_2(s))|\,ds \\
&\leq \frac{k\,|u_1(s) - u_2(s)|}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{|b|\,k\,|u_1(s) - u_2(s)|}{\Gamma(\alpha)\,|a+b|}\int_0^T (T-s)^{\alpha-1}ds \\
&\leq \frac{kT^{\alpha}\left(1 + \frac{|b|}{|a+b|}\right)}{\alpha\Gamma(\alpha)}\,|u_1(s) - u_2(s)|\,.
\end{aligned}$$

So

$$\|Fu_1 - Fu_2\|_{\infty} \leq \frac{kT^{\alpha}\left(1 + \frac{|b|}{|a+b|}\right)}{\Gamma(\alpha+1)}\,\|u_1 - u_2\|_{\infty}\,.$$

By virtue of $\frac{kT^{\alpha}\left(1+\frac{|b|}{|a+b|}\right)}{\Gamma(\alpha+1)} < 1$, we can deduce that F is a contraction and according to Banach's theorem, F admits a unique fixed point which is a solution of problem (4.3.1). ■

We use Schaefer's fixed point theorem as the second result of the solution of (4.3.1).

Theorem 4.3.2 *Consider the following two conditions:*

(H1) The function $f : [0, T] : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous.*

(H2) There exists a constant $M > 0$ *such that*

$$|f(t, u)| \leq M, \forall t \in [0, T] \text{ and } u \in \mathbb{R}.$$

Then, problem (4.3.1) admits at least one solution on $[0, T]$.

Proof. We will use Schaefer's fixed point theorem to show that F defined by (4.3.4) admits a fixed point. The demonstration is done in several stages.

Step 1. F is continuous.

Let $\{u_n\}$ be a sequence in $C([0, T], \mathbb{R})$ converge for $\|.\|_\infty$ to u, i.e.

$$\lim_{n \longrightarrow \infty} \|u_n - u\|_\infty = 0.$$

It must be shown that $\lim_{n \longrightarrow \infty} \|Fu_n - Fu\|_\infty = 0. \forall t \in [0, T]$, we have

$$\begin{aligned}
|Fu_n(t) - Fu(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, u_n(s)) - f(s, u(s))| \, ds \\
&\quad + \frac{|b|}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} |f(s, u_n(s)) - f(s, u(s))| \, ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sup_{s \in [0,T]} |f(s, u_n(s)) - f(s, u(s))| \, ds \\
&\quad + \frac{|b|}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} \sup_{s \in [0,T]} |f(s, u_n(s)) - f(s, u(s))| \, ds \\
&\leq \frac{\|f(., u_n(.)) - f(., u(.))\|_\infty}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} ds + \frac{|b|}{|a+b|} \int_0^T (T-s)^{\alpha-1} ds \right] \\
&\leq \frac{T^\alpha \left(1 + \frac{|b|}{|a+b|}\right)}{\alpha \Gamma(\alpha)} \|f(., u_n(.)) - f(., u(.))\|_\infty .
\end{aligned}$$

Since f is continuous, then

$$\|Fu_n - Fu\|_\infty \leq \frac{T^\alpha \left(1 + \frac{|b|}{|a+b|}\right)}{\alpha \Gamma(\alpha)} \|f(., u_n(.)) - f(., u(.))\|_\infty \rightarrow 0 \text{ when } n \longrightarrow \infty.$$

Hence the continuity of F.

Step 2. The image of any set bounded by F is a set bounded in $C([0,T],\mathbb{R})$, in fact it is enough to show that for all $r>0$, there exists a constant $L>0$, for all $u\in B_r$

$$B_r=\{u\in C([0,T],\mathbb{R}),\|u\|_\infty\leq r\},$$

we have $\|Fu\|_\infty\leq L$.

By (H2) we have for all $t\in[0,T]$,

$$\begin{aligned}|Fu(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}|f(s,u(s))|\,ds+\frac{|b|}{\Gamma(\alpha)\,|a+b|}\int_0^T(T-s)^{\alpha-1}|f(s,u(s))|\,ds+\frac{|c|}{|a+b|}\\ &\leq \frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{M\,|b|}{\Gamma(\alpha)\,|a+b|}\int_0^T(T-s)^{\alpha-1}ds+\frac{|c|}{|a+b|}\\ &\leq \frac{M}{\alpha\Gamma(\alpha)}T^\alpha+\frac{M\,|b|}{\alpha\Gamma(\alpha)\,|a+b|}T^\alpha+\frac{|c|}{|a+b|}.\end{aligned}$$

So

$$\|Fu\|_\infty\leq\frac{M}{\Gamma(\alpha+1)}T^\alpha+\frac{M\,|b|}{\Gamma(\alpha+1)\,|a+b|}T^\alpha+\frac{|c|}{|a+b|}=L.$$

Consequently $F(B_r)$ is bounded.

Step 3. The image of everything bounded by F is an equicontinuous set of $C([0,T],\mathbb{R})$. Let $t_1,t_2\in(0,T],t_1<t_2,B_r$ a bounded set of $C([0,T],\mathbb{R})$ and let $u\in B_r$, so

$$\begin{aligned}|Fu(t_1)-Fu(t_2)| &\leq \frac{M}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}\right]ds\\ &\quad+\frac{M}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\\ &\leq \frac{M}{\alpha\Gamma(\alpha)}\left[(t_2-t_1)^\alpha+t_1^\alpha-t_2^\alpha\right]+\frac{M}{\alpha\Gamma(\alpha)}(t_2-t_1)^\alpha\\ &\leq \frac{M}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha+\frac{M}{\Gamma(\alpha+1)}(t_1^\alpha-t_2^\alpha).\end{aligned}$$

when $t_1\longrightarrow t_2$, momber right of the previous inequality tends towards 0, hence the continuity of F according to step 2 and 3 and the Ascoli-Arzélà theorem, $F(B_r)$ is relatively compact for all bounded B_r. i.e. F is completely continuous and by step 1, F is continuous hence, $F:C([0,T],\mathbb{R})\longrightarrow C([0,T],\mathbb{R})$ is continuous and completely continuous.

Step 4. So it remains to show that $\varepsilon=\{u\in C(A,\mathbb{R}):u=\lambda Fu\}$ such that $0<\lambda<1$, is limited.

Let $u\in\varepsilon\Longrightarrow u=\lambda Fu$, therefore $\forall t\in[0,T]$, we have

$$u=\lambda\left[\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,u(s))ds-\frac{1}{a+b}\left[\frac{b}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f(s,u(s))ds-c\right]\right],$$

and according to (H2) and $\forall t \in [0, T]$, we have

$$\begin{aligned} |Fu(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s,u(s))| \, ds + \frac{|b|}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} |f(s,u(s))| \, ds + \frac{|c|}{|a+b|} \\ &\leq \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds + \frac{M |b|}{\Gamma(\alpha) |a+b|} \int_0^T (T-s)^{\alpha-1} ds + \frac{|c|}{|a+b|} \\ &\leq \frac{M}{\alpha\Gamma(\alpha)} t^\alpha + \frac{M |b|}{\alpha\Gamma(\alpha) |a+b|} T^\alpha + \frac{|c|}{|a+b|}. \end{aligned}$$

So

$$\|Fu\|_\infty \leq \frac{M}{\Gamma(\alpha+1)} T^\alpha + \frac{M |b|}{\Gamma(\alpha+1) |a+b|} T^\alpha + \frac{|c|}{|a+b|} = R.$$

This shows that ε is bounded. then according to Schaefer's theorem 4.1.5 we deduce that F admits at least one fixed point that is a solution of the problem (4.3.1) on $[0, T]$. ■