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Chapter 2 : Random Variables -Continuous random variables.

1. Continuous random variables.

A continuous random variable can be defined as a random variable that can take on an infinite number of possible values. It is used to denote measurements such as height, weight, time, etc. Due to this, the probability that a continuous random variable will take on an exact value is 0. The cumulative distribution function (CDF) and the probability density function (PDF) are used to describe the characteristics of a continuous random variable. The main difference between continuous and discrete random variables is that continuous probability is measured over intervals, while discrete probability is calculated on exact points.

Exemple 1. - The time it takes to complete an exam for a 60 minute test. Possible values = all real numbers on the interval $[0, 60]$.
- Age of a fossil. Possible values = all real numbers on the interval [minimum age, maximum age].
- Miles per gallon for a such car. Possible Values = all real numbers on the interval [minimum MPG, maximum MPG].

Définition 1. A random variable X is said to be continuous if and only if the probability that it will belong to an interval $[a, b]$ can be expressed as an integral :

$$P(X \in [a, b]) = \int_a^b f_X(x)dx$$

where the integrand function $f_X(x) : \mathbb{R} \to [0, +\infty)$ is called the probability density function (PDF) of X.

Exemple 2. Let X be a continuous random variable that can take any value in the interval $[0, 1]$.

$$f_X(x) = \begin{cases} 4x^3 & \text{if } x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases}$$

The probability that X takes a value between 0.5 and 1 can be computed as follows :

$$\begin{aligned} P(X \in [0.5, 1]) &= \int_{0.5}^1 f_X(x)dx \\ &= \int_{0.5}^1 4x^3dx = \frac{15}{16} \end{aligned}$$

1.1. Probability density function (PDF)

Recall that continuous random variables have uncountably many possible values (think of intervals of real numbers). Just as for discrete random variables, we can talk about probabilities for continuous random variables using density functions.

Définition 2. The probability density function (PDF), denoted f_X, of a continuous random variable X satisfies the following properties :

1- $f_X(x) \geq 0$, for all $x \in \mathbb{R}$.
2- f_X is piecewise continuous.
3- $\int_{\mathbb{R}} f(x)\, dx = 1$
4- $P(a \leq X \leq b) = \int_a^b f(x)\, dx$

Note that, unlike discrete random variables, continuous random variables have zero point probabilities, i.e., the probability that a continuous random variable equals a single value is always given by 0. Formally, this follows from properties of integrals :

$$P(X = a) = P(a \leq X \leq a) = \int_a^a f(x)\,dx = 0.$$

1.2. ***Cumulative distribution function (CDF)***

Recall Definition **??**, the definition of the CDF, which applies to both discrete and continuous random variables. For continuous random variables we can further specify how to calculate the CDF with a formula as follows.

Définition 3. Let X be a continuous random variable that has PDF f_X, then the CDF F_X is given by

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt, \quad \text{for } x \in \mathbb{R}.$$

In other words, the CDF for a continuous random variable is found by integrating its PDF.

1.3. ***Mean and standard deviation of continuous random variables***

Définition 4. If X is a continuous random variable with PDF f_X, then the mean of X is given by

$$\mu = \mu_X = \mathrm{E}[X] = \int_{\mathbb{R}} x \cdot f_X(x)\,dx.$$

- For the variance of a continuous random variable, the definition is the same as given by Definition **??**, only we now integrate to calculate the value :

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mu^2 = \left(\int_{\mathbb{R}} x^2 \cdot fX(x)\,dx \right) - \mu^2$$

Exemple 3. Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{3}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

Solution

$$\begin{aligned} E[X] &= \int_{\mathbb{R}} x f_X(x) dx \\ &= \int_1^{+\infty} \frac{3}{x^3} dx \\ &= \left[-\frac{3}{2} x^{-2} \right]_1^{+\infty} = \frac{3}{2} \end{aligned}$$

Then, $\mu_X = 3/2$. Next, we have

$$\begin{aligned} E[X^2] &= \int_{\mathbb{R}} x^2 f_X(x) dx \\ &= \int_1^{+\infty} \frac{3}{x^2} dx \\ &= \left[-3x^{-1} \right]_1^{+\infty} = 3 \end{aligned}$$

Thus, we have

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 3 - \frac{9}{4} = \frac{3}{4}.$$

1.4. Some continuous distributions

1.4.1. Uniform distribution

The uniform distribution is a continuous probability distribution and is concerned with events that are equally likely to occur.

Définition 5. A random variable X is governed by the Uniform distribution with parameters a and b, noted as $X \sim U(a,b)$, if its density function is

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

Proposition 1. Let $X \sim U(a,b)$, then

$$\mathbb{E}(X) = \frac{a+b}{2}$$
$$Var(X) = \frac{(b-a)^2}{12}$$

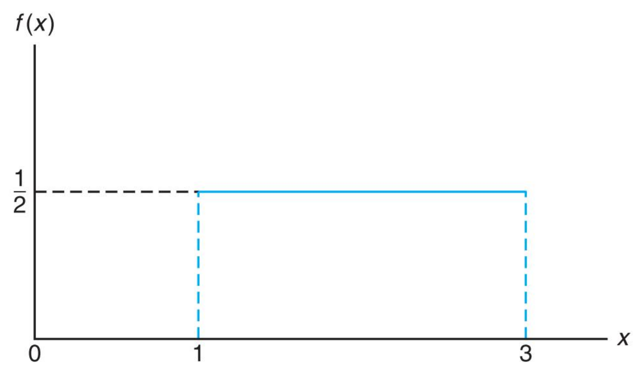


FIGURE 1: The density function for a uniform random variable on the interval $[1,3]$.

Exemple 4. Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0,4]$.
a- What is the probability density function ?
b- What is the probability that any given conference lasts at least 3 hours ?

Solution

a- The appropriate density function for the uniformly distributed random variable X in this situation is

$$f_X(x) = \begin{cases} \frac{1}{4} & x \in [0,4] \\ 0 & \text{otherwise} \end{cases}$$

b-

$$P(X \geq 3) = \int_3^4 \frac{1}{4}\,dx = \frac{1}{4}.$$

1.4.2. ***Normal (Gaussian) distribution***

The normal distribution, which is continuous, is the most important of all the probability distributions. Its graph is bell-shaped, see Figure 2. This bell-shaped curve is used in almost all disciplines. Since it is a continuous distribution, the total area under the curve is one. The parameters of the normal are the mean μ and the standard deviation σ. A special normal distribution, called the standard normal distribution is the distribution of z-scores. Its mean is **zero**, and its standard deviation is **one**.

Définition 6. A random variable X is governed by the Normal(Gaussian) distribution with parameters μ (mean) and σ (standard deviation), noted as $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

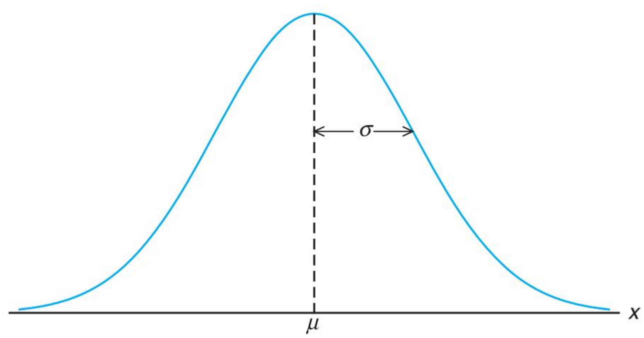


FIGURE 2: The density function for a Normal random variable.

Figure 3 shows two normal curves having different means and different standard deviations. Clearly, they are centred at different positions on the horizontal axis and their shapes reflect the two different values of σ.

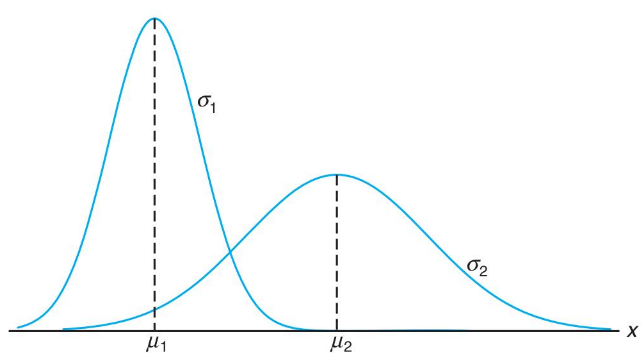


FIGURE 3: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

Proposition 2. Let $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$\mathbb{E}(X) = \mu$$
$$Var(X) = \sigma^2$$

Définition 7. The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

Théorème 1. *If Z is a standard normal random variable i.e., $Z \sim \mathcal{N}(0,1)$ and $X = \sigma Z + \mu$, then X is a normal random variable with mean μ and variance σ^2, i.e,*

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

Conversely, if $X \sim \mathcal{N}(\mu, \sigma^2)$, the random variable defined by $Z = \frac{X-\mu}{\sigma}$ is a standard normal random variable, i.e., $Z \sim \mathcal{N}(0,1)$.

Démonstration. Let $\Phi(.)$ the CDF of Z, then to compute the CDF of X, we can write

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(\sigma Z + \mu \leq x) \qquad \big(\text{where } Z \sim \mathcal{N}(0,1)\big) \\ &= P\left(Z \leq \frac{x-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{x-\mu}{\sigma}\right). \end{aligned}$$

It results that

$$F_X(x) = P(X \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

and

$$P(a \leq X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

□

Exemple 5. Let $X \sim \mathcal{N}(-5, 4)$. Find the probabilities
- a- $P(X < 0)$
- b- $P(-7 < X < -3)$
- c- $P(X > -3 | X > -5)$

Solution
Since $\mu = -5$ and $\sigma = 2$, then
a-

$$\begin{aligned} P(X < 0) &= F_X(0) \\ &= \Phi\left(\frac{0-(-5)}{2}\right) \\ &= \Phi(2.5) \approx 0.99 \end{aligned}$$

b-

$$\begin{aligned} P(-7 < X < -3) &= F_X(-3) - F_X(-7) \\ &= \Phi\left(\frac{(-3)-(-5)}{2}\right) - \Phi\left(\frac{(-7)-(-5)}{2}\right) \\ &= \Phi(1) - \Phi(-1) \\ &= 2\Phi(1) - 1 \qquad \big(\text{since } \Phi(-x) = 1 - \Phi(x)\big) \approx 0.68 \end{aligned}$$

c-

$$\begin{aligned} P(X > -3 | X > -5) &= \frac{P(X > -3, X > -5)}{P(X > -5)} \\ &= \frac{P(X > -3)}{P(X > -5)} \\ &= \frac{1 - \Phi\left(\frac{(-3)-(-5)}{2}\right)}{1 - \Phi\left(\frac{(-5)-(-5)}{2}\right)} \\ &= \frac{1 - \Phi(1)}{1 - \Phi(0)} \\ &\approx \frac{0.1587}{0.5} \approx 0.32 \end{aligned}$$

Théorème 2. *If* $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$*, and* $Y = aX + b$*, where* $a, b \in \mathbb{R}$*, then* $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ *where*

$$\mu_Y = a\mu_X + b, \quad \sigma_Y^2 = a^2\sigma_X^2.$$

Démonstration. Since we have

$$X = \sigma_X Z + \mu_X \qquad \text{where } Z \sim \mathcal{N}(0, 1),$$

it follows that

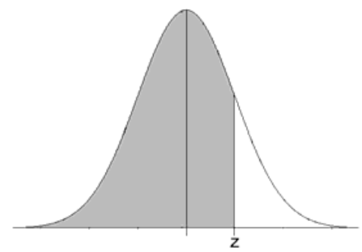
$$\begin{aligned} Y &= a(\sigma_X Z + \mu_X) + b \\ &= (a\sigma_X)Z + (a\mu_X + b). \end{aligned}$$

Then

$$Y \sim \mathcal{N}(a\mu_X + b, a^2\sigma_X^2).$$

□

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1.4.3. ***Exponential distribution***

Exponential distribution is often used to predict the waiting time until the next event occurs, such as a success, failure, or arrival. For example the amount of time you need to wait for the bus to arrive.

Définition 8. A random variable X is governed by the exponential distribution with parameter $\lambda > 0$, noted as $X \sim \exp(\lambda)$, if its density function is

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & if \quad x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Figure 4 shows the PDF of the exponential distribution for some values of λ.

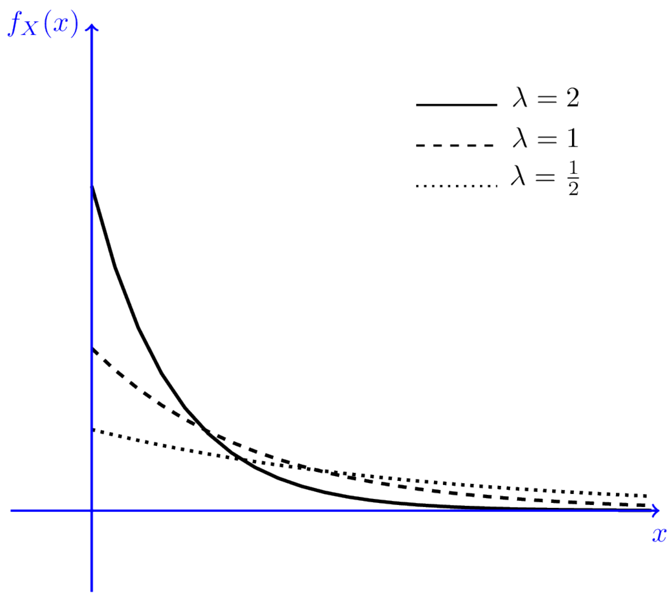


Figure 4: Probability density functions of the exponential distribution for some values of λ.

Proposition 3. Let $X \sim \exp(\lambda), \lambda > 0$, then

$$\mathbb{E}(X) = \lambda$$
$$Var(X) = \frac{1}{\lambda^2}$$

Exemple 6. Assume that, you usually get 2 phone calls per hour. calculate the probability, that a phone call will come within the first hour.

Solution

It is given that, 2 phone calls per hour. Then, it would expect that one phone call at every half-an-hour. So, we can take $\lambda = 0.5$. Hence, the computation is as follows :

$$p(0 \leq X \leq 1) = \int_0^1 0.5e^{-0.5x}$$

Therefore, the probability of arriving the phone calls within the first hour is 0.393

1.4.4. ***Gamma distribution***

Gamma distributions are usually used to model wait times until events. Unlike the exponential distribution which models the time until the first event, gamma models the time until the α-th event. Before introducing the gamma random variable, we need to introduce the gamma function.

Définition 9. Let us take a parameter $\alpha > 0$. Gamma function $\Gamma(\alpha)$ is an extension of the factorial function to real (and complex) numbers i.e., for $\alpha \in \{1, 2, 3, \ldots\}$, then

$$\Gamma(\alpha) = (\alpha - 1)!$$

More generally, for any positive real number α the gamma function is defined as follows

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$$

Proposition 4. Let α a strictly positive real number, then gamma function has the following properties :

- $\displaystyle\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\Gamma(\alpha)}{\lambda^{\alpha}}$, for $\lambda > 0$;
- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$;
- $\Gamma(\alpha) = (\alpha - 1)!$, for $\alpha = 1, 2, 3, \cdots$;

Now we define the gamma distribution by its PDF

Définition 10. A random variable X is governed by the gamma distribution with parameters $\alpha > 0$ and $\lambda > 0$, noted as $X \sim Gamma(\alpha, \lambda)$, if its density function is

$$f_X(x) = \begin{cases} \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Figure 5 shows the PDF of Gamma distribution for some values of λ and α.

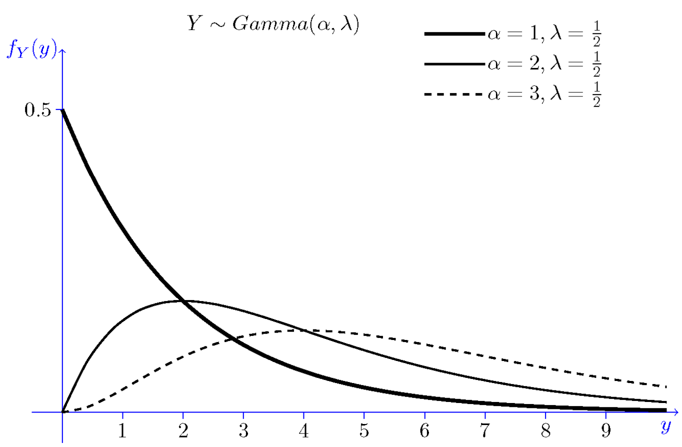


FIGURE 5: Probability density functions of the gamma distribution for some values of λ and α.

Proposition 5. Let $X \sim Gamma(\alpha, \lambda)$, then

$$\mathbb{E}(X) = \frac{\alpha}{\lambda}$$
$$Var(X) = \frac{\alpha}{\lambda^2}.$$

Exemple 7. On average, someone sends a money order once per our. What is the probability someone sends 3 money orders in less than 3 hours ?.