# Chapter 3

# Fractional differential equations

In this chapter, we present the different types of fractional differential equations (FDEs) in particular, the Riemann-Liouville type fractional differential equation and the Caputo type fractional differential equation.

**Definition 3.0.9** *Let $\alpha > 0, \alpha \notin \mathbb{N}$, $n = [\alpha] + 1$ and $f : A \subset \mathbb{R}^2 \to \mathbb{R}$, then*

$$D^{\alpha}u(t) = f(t, u(t)), \tag{3.0.1}$$

*is called a fractional differential equation of the Riemann-Liouville type. The initial conditions for this type of FDE, we use*

$$D^{\alpha-k}u(0) = b_k, k = 0, 1, 2, ..., n-1, \lim_{t\to 0} I^{n-\alpha}u(t) = b_n.$$

*In the same way*

$${}^{C}D^{\alpha}u(t) = f(t, u(t)), \tag{3.0.2}$$

*is called a fractional differential equation of the Caputo type, and in this case we use the initial conditions as*

$$u^{(k)}(0) = b_k, k = 0, 1, 2, ..., n-1.$$

The use of initial conditions of different types for the fractional differential equations (3.0.1) and (3.0.2) ensures the uniqueness of the solutions of the corresponding FDE, which we will prove in the following theorems

## 3.1 Fractional differential equation of the Riemann-Liouville type

We start with the homogeneous equation of Riemann-Liouville type.

**Lemma 3.1.1** *Let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation of the Riemann-Liouville type*

$$D^{\alpha}_{0^+} u(t) = 0, 0 < t < 1, \tag{3.1.1}$$

*admits a unique solution*

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n},$$

*where $C_m \in \mathbb{R}$ with $m = 1, 2, ..., n$.*

**Proof.** Let $\alpha > 0$. According to Remark 2.2.2, we have

$$D^{\alpha}_{0^+} t^{\alpha-m} = 0 \text{ with } m = 1, 2, ..., n.$$

Then the fractional differential equation (3.1.1), admits a particular solution, as

$$u(t) = C_m t^{\alpha-m} \text{ with } m = 1, 2, ..., n, \tag{3.1.2}$$

where $C_m \in \mathbb{R}$.

So the general solution of (3.1.1) given as a sum of particular solutions (3.1.2) i.e.

$$u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n},$$

where $C_m \in \mathbb{R}$ with $m = 1, 2, ..., n$. ■

**Lemma 3.1.2** *Suppose that*

$$u \in C(0,1) \cap L^1(0,1) \textit{ and } D^{\alpha}_{0^+} u \in C(0,1) \cap L^1(0,1).$$

*Then*

$$I^{\alpha}_{0^+} D^{\alpha}_{0^+} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + ... + C_n t^{\alpha-n}, \tag{3.1.3}$$

*where $C_m \in \mathbb{R}$ with $m = 1, 2, ..., n$.*

**Proof.** Let $\alpha > 0$. For all $u \in C(0,1) \cap L^1(0,1)$ (Proposition 2.2.3) we have

$$\begin{aligned} I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) &= u(t) - \sum_{k=0}^{n} \frac{I_{0^+}^{n-\alpha} u^{(n-k)}(0)}{\Gamma(\alpha - k + 1)} t^{\alpha - k} \\ &= u(t) - \left[ \frac{I_{0^+}^{n-\alpha} u^{(n-1)}(0)}{\Gamma(\alpha)} t^{\alpha-1} + \frac{I_{0^+}^{n-\alpha} u^{(n-2)}(0)}{\Gamma(\alpha-1)} t^{\alpha-2} + ... + \frac{I_{0^+}^{n-\alpha} u^{(n-n)}(0)}{\Gamma(\alpha-n+1)} t^{\alpha-n} \right]. \end{aligned}$$

We put $C_m = -\frac{I_{0^+}^{n-\alpha} u^{(n-m)}(0)}{\Gamma(\alpha-m+1)} \in \mathbb{R}$ for each $m = 1, 2, ..., n$, we find the equality (3.1.3). ■

**Lemma 3.1.3** *Let $1 < \alpha \leq 2$ and $y \in C(0,1)$.*

*So the unique solution to the boundary problem*

$$\begin{cases} D_{0^+}^{\alpha} u(t) + y(t) = 0, 0 < t < 1 \\ u(0) = u(1) = 0 \end{cases}, \tag{3.1.4}$$

*is given by*

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

*such as*

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} & si\ 0 \leq s \leq t \leq 1 \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} & si\ 0 \leq t \leq s \leq 1 \end{cases}.$$

**Proof.** By applying $I_{0^+}^{\alpha}$ to equation (3.1.4) we obtain

$$I_{0^+}^{\alpha} \left[ D_{0^+}^{\alpha} u(t) + y(t) \right] = 0 \Longleftrightarrow I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) + I_{0^+}^{\alpha} y(t) = 0.$$

According to Lemma 3.1.2 for $1 < \alpha \leq 2$ $(n = [\alpha] + 1 = 2)$ we have

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}, C_1, C_2 \in \mathbb{R},$$

so

$$u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + I_{0^+}^{\alpha} y(t) = 0,$$

which implies

$$u(t) = -I_{0^+}^{\alpha} y(t) - C_1 t^{\alpha-1} - C_2 t^{\alpha-2}.$$

Therefore, the general solution of equation (3.1.4) given by

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - C_1 t^{\alpha-1} - C_2 t^{\alpha-2}. \tag{3.1.5}$$

The boundary conditions imply that

$$\begin{cases} u(0)=0 \Longrightarrow 0=-0-0-\lim\limits_{t\to 0} C_2 t^{\alpha-2} \\ u(0)=1 \Longrightarrow 0=-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds - C_1 \end{cases} \Longrightarrow \begin{cases} C_2=0 \\ C_1=-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \end{cases}.$$

The integro-differential equation (3.1.5), equivalent to

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}y(s)ds \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

■

## 3.2 Fractional differential equation of the Caputo type

We start with the homogeneous equation of Caputo type.

**Lemma 3.2.1** *Let $\alpha > 0$. If we assume that $u \in C(0,1)\cap L^1(0,1)$, then the fractional differential equation of the Caputo type*

$$^{C}D^{\alpha}_{0^+}u(t)=0, 0<t<1, \tag{3.2.1}$$

*admits a unique solution*

$$u(t)=C_0+C_1t+C_2t^2+...+C_{n-1}t^{n-1},$$

*where $C_m \in \mathbb{R}$ with $m=0,1,2,...,n-1$.*

**Proof.** Let $\alpha > 0$. According to Remark 2.2.5, we have

$$D^{\alpha}_{0^+}t^m = 0 \text{ with } m=0,1,2,...,n-1.$$

Then the fractional differential equation (3.2.1), admits a particular solution, as

$$u(t)=C_mt^m \text{ with } m=0,1,2,...,n-1, \tag{3.2.2}$$