The boundary conditions imply that

$$\begin{cases} u(0)=0 \Longrightarrow 0=-0-0-\lim\limits_{t\to 0} C_2 t^{\alpha-2} \\ u(0)=1 \Longrightarrow 0=-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds - C_1 \end{cases} \Longrightarrow \begin{cases} C_2=0 \\ C_1=-\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \end{cases}.$$

The integro-differential equation (3.1.5), equivalent to

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds \\ &= -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t (1-s)^{\alpha-1}y(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_t^1 (1-s)^{\alpha-1}y(s)ds \\ &= \int_0^t \frac{[t(1-s)]^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds + \int_t^1 \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds. \end{aligned}$$

■

3.2 Fractional differential equation of the Caputo type

We start with the homogeneous equation of Caputo type.

Lemma 3.2.1 *Let $\alpha > 0$. If we assume that $u \in C(0,1) \cap L^1(0,1)$, then the fractional differential equation of the Caputo type*

$$^{C}D^{\alpha}_{0+}u(t) = 0, 0 < t < 1, \tag{3.2.1}$$

admits a unique solution

$$u(t) = C_0 + C_1 t + C_2 t^2 + ... + C_{n-1}t^{n-1},$$

where $C_m \in \mathbb{R}$ with $m = 0, 1, 2, ..., n-1$.

Proof. Let $\alpha > 0$. According to Remark 2.2.5, we have

$$D^{\alpha}_{0+}t^m = 0 \text{ with } m = 0, 1, 2, ..., n-1.$$

Then the fractional differential equation (3.2.1), admits a particular solution, as

$$u(t) = C_m t^m \text{ with } m = 0, 1, 2, ..., n-1, \tag{3.2.2}$$

where $C_m \in \mathbb{R}$.

So the general solution of (3.2.1) given as a sum of particular solutions (3.2.2) i.e.

$$u(t) = C_0 + C_1 t + C_2 t^2 + ... + C_{n-1} t^{n-1},$$

where $C_m \in \mathbb{R}$ with $m = 0, 1, 2, ..., n-1$. ■

Lemma 3.2.2 *Suppose that $u \in C^n([0,1])$*

Then

$$I_{0^+}^{\alpha}\ {}^{C}D_{0^+}^{\alpha} u(t) = u(t) + C_0 + C_1 t + C_2 t^2 + ... + C_{n-1} t^{n-1}, \tag{3.2.3}$$

where $C_m \in \mathbb{R}$ with $m = 0, 1, 2, ..., n-1$.

Proof. Let $\alpha > 0$. For all $u \in C^n([0,1])$ (Proposition 2.2.4) we have

$$\begin{aligned} I_{0^+}^{\alpha}\ {}^{C}D_{0^+}^{\alpha} u(t) &= u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k \\ &= u(t) - \left[u(0) + u'(0)t + \frac{u''(0)}{2!} t^2 + ... + \frac{u^{(n-1)}(0)}{k!} t^{n-1} \right]. \end{aligned}$$

We put $C_m = -\frac{u^{(m)}(0)}{m!} \in \mathbb{R}$ for each $m = 1, 2, ..., n$, we find the equality (3.2.3). ■

Lemma 3.2.3 *Let $1 < \alpha \leq 2$ and $y \in C(0,1)$.*

So the unique solution to the boundary problem

$$\begin{cases} {}^{C}D_{0^+}^{\alpha} u(t) = y(t), 0 < t < 1 \\ u(0) + u'(0) = 0, u(1) + u'(1) = 0 \end{cases}, \tag{3.2.4}$$

is given by

$$u(t) = \int_0^1 G(t,s) y(s) ds,$$

such as

$$G(t,s) = \begin{cases} \frac{(1-t)(1-s)^{\alpha-1} + (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & si\ 0 \leq s \leq t \leq 1 \\ \frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} & si\ 0 \leq t \leq s \leq 1 \end{cases}. \tag{3.2.5}$$

Proof. By applying $I_{0^+}^{\alpha}$ to equation (3.2.4) we obtain

$$I_{0^+}^{\alpha} \left[{}^{C}D_{0^+}^{\alpha} u(t) - y(t) \right] = 0 \Longleftrightarrow I_{0^+}^{\alpha}\ {}^{C}D_{0^+}^{\alpha} u(t) - I_{0^+}^{\alpha} y(t) = 0.$$

According to Lemma 3.2.2 for $1 < \alpha \leq 2$ ($n = [\alpha] + 1 = 2$) we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_0 + C_1 t, C_0, C_1 \in \mathbb{R},$$

so

$$u(t) + C_0 + C_1 t - I_{0+}^{\alpha} y(t) = 0,$$

which implies

$$u(t) = I_{0+}^{\alpha} y(t) - C_0 - C_1 t.$$

Therefore, the general solution of equation (3.2.4) given by

$$u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds - C_0 - C_1 t. \tag{3.2.6}$$

The boundary conditions imply that

$$\begin{cases} u(0) + u'(0) = 0 \\ u(1) + u'(1) = 0 \end{cases} \Longrightarrow \begin{cases} C_0 + C_1 = 0 \\ C_0 + 2C_1 = \left(I_{0+}^{\alpha} y\right)(1) + \left(I_{0+}^{\alpha} y\right)'(1) \end{cases}.$$

So

$$\begin{cases} C_0 = -\left(I_{0+}^{\alpha} y\right)(1) - \left(I_{0+}^{\alpha} y\right)'(1) \\ = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\ C_1 = \left(I_{0+}^{\alpha} y\right)(1) + \left(I_{0+}^{\alpha} y\right)'(1) \\ = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \end{cases}$$

The integro-differential equation (3.2.6), equivalent to

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\ &\quad - \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s) ds - \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} y(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds + \frac{(1-t)}{\Gamma(\alpha)} \int_0^t (1-s)^{\alpha-1} y(s) ds + \frac{(1-t)}{\Gamma(\alpha)} \int_t^1 (1-s)^{\alpha-1} y(s) ds \\ &\quad + \frac{(1-t)}{\Gamma(\alpha-1)} \int_0^t (1-s)^{\alpha-2} y(s) ds + \frac{(1-t)}{\Gamma(\alpha-1)} \int_t^1 (1-s)^{\alpha-2} y(s) ds \\ &= \int_0^t \left[\frac{(1-t)(1-s)^{\alpha-1} + (t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right] y(s) ds \\ &\quad + \int_t^1 \left[\frac{(1-t)(1-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{(1-t)(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right] y(s) ds = \int_0^1 G(t,s) y(s) ds. \end{aligned}$$

■