

### Chapter 3 : Probability distributions for combined random variables : Joint distributions of continuous random variables

#### 1. Joint distributions of continuous random variables

As considered in the discrete case, we interest now by the joint distributions for continuous case.

**Définition 1. (joint probability density function)** Let  $X$  and  $Y$  are two continuous random variables that defined on the same sample space  $S$ , then their joint probability density function is a piecewise continuous function, denoted  $f(x, y)$ , that satisfies the following properties.

- 1-  $f(x, y) \geq 0$ , for all  $(x, y) \in \mathbb{R}^2$
- 2-  $\iint_{\mathbb{R}^2} f(x, y) dx dy = 1$
- 3-  $P((X, Y) \in A) = \iint_A f(x, y) dx dy$ , for any  $A \subseteq \mathbb{R}^2$

**Définition 2. (joint cumulative distribution function)** Let  $X$  and  $Y$  are continuous random variables that are defined on the same sample space  $S$ , then their joint cumulative distribution function is obtained by integrate the joint PDF i.e, if  $A$  is given as

$$A = \{(x, y) \in \mathbb{R}^2 \mid X \leq a \text{ and } Y \leq b\}$$

where  $a$  and  $b$  are constants, then the joint CDF of  $X$  and  $Y$ , at the point  $(a, b)$ , is given by

$$F(a, b) = P(X \leq a, Y \leq b) = \int_{-\infty}^b \int_{-\infty}^a f(x, y) dx dy$$

Similarly to the discrete case, we can derive the individual marginal probability density functions (PDF's) of  $X$  and  $Y$  from the joint PDF.

**Définition 3.** Let  $X$  and  $Y$  are continuous random variables and let  $f(x, y)$  their joint PDF. The marginal probability density functions of  $X$  and  $Y$  are respectively defined as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (\text{fix a value of } X, \text{ and integrate over all possible values of } Y)$$
$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad (\text{fix a value of } Y, \text{ and integrate over all possible values of } X)$$

Similarly, in the continuous case, we can also define independent random variables in the same manner as we did for discrete random variables.

**Définition 4. (Independent random variables)**

Continuous random variables  $X_1, X_2, \dots, X_n$  are considered independent if the joint probability density function factors into a product of the marginal PDF's i.e,

$$f(x_1, x_2, \dots, x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

This condition holds also for the cumulative distribution functions.

1.1. Expectations of joint continuous distributions

Let  $X$  and  $Y$  be continuous random variables and let  $f(x, y)$  be their joint PDF. If  $g(X, Y)$  is a function of these two random variables, then its mean value is given as follows :

$$E[g(X, Y)] = \iint_{\mathbb{R}^2} g(x, y)f(x, y)dxdy$$

**Example 1.** Let  $X$  and  $Y$  be two joint continuous random variables having the following joint PDF

$$f_{XY}(x, y) = \begin{cases} x + y & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

- Determine  $E[XY^2]$

**Solution**

$$\begin{aligned} E[XY^2] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy^2 f_{XY}(x, y) dx dy \\ &= \int_0^1 \int_0^1 xy^2(x + y) dx dy \\ &= \int_0^1 \int_0^1 x^2y^2 + xy^3 dx dy \\ &= \int_0^1 \left( \frac{1}{3}y^2 + \frac{1}{2}y^3 \right) dy \\ &= \frac{17}{72}. \end{aligned}$$