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Solved exercises

Exercise 1 Assume that a radioactive particle is confined in a unit square and X and Y are the random variables, which represent the particle's location in the unit square, with the bottom left corner placed at the origin. Radioactive particles follow entirely random behavior that is uniformly distributed over the unit square, having the following joint PDF

$$f(x,y) = \begin{cases} c, & \text{if } 0 \leq x, y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where c is a constant.
1- Determine the value of the constant c.
2- Let the set $A = \{(x,y) \mid x - y > 0.5\}$. Find the probability $P((X,Y) \in A)$
3- Find the marginal PDF's of X and Y
4- Determine $E[XY]$.
Solution
1- From joint PDF definition, it results that $\iint_{\mathbb{R}^2} f(x,y)dxdy = 1$, therefore,

$$\int_0^1 \int_0^1 c dxdy = 1 \quad \Rightarrow \quad c \int_0^1 \int_0^1 1 dxdy = 1 \quad \Rightarrow \quad c = 1$$

2- Figure 1 represents the graph of region A, then

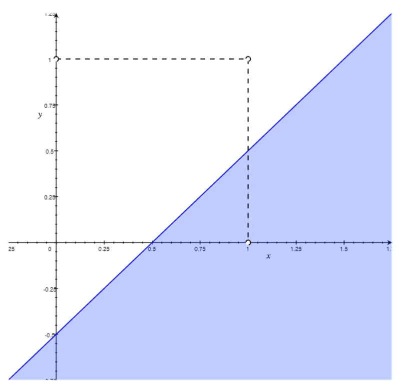

Figure 1: Graph of region A, shaded in blue.

$$P(X - Y > 0.5) = \iint_A f(x,y)dxdy = \int_0^{0.5} \int_{y+0.5}^1 1 dxdy = 0.125$$

3-

$$f_X(x) = \int_0^1 1 dy = 1, \quad \text{for } 0 \leq x \leq 1$$
$$f_Y(y) = \int_0^1 1 dx = 1, \quad \text{for } 0 \leq y \leq 1$$

4-The expected value of XY

$$\mathrm{E}[XY] = \iint_{\mathbb{R}^2} xy \cdot f(x,y)dxdy = \int_0^1 \int_0^1 xy \cdot 1 dxdy = \int_0^1 \left(\frac{x^2}{2} y \Big|_0^1 \right) dy = \frac{1}{4}$$

Exercise 2 Let $X, Y \sim U(0,1)$ and they are independents. Find the cumulative joint distribution.

Solution

We have

$$F_X(x) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

and

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \le y \le 1 \\ 1 & \text{for } y > 1 \end{cases}$$

Since X and Y are independent, then

$$F_{XY}(x,y) = F_X(x)F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \text{ or } x < 0 \\ xy & \text{for } 0 \le x, y \le 1 \\ y & \text{for } x > 1, 0 \le y \le 1 \\ x & \text{for } y > 1, 0 \le x \le 1 \\ 1 & \text{for } x > 1, y > 1 \end{cases}$$

Exercise 2 Two points are chosen uniformly and independently along a stick of length 1. determine the mean distance between those two points ?

Solution

We aim to find $E[|X-Y|]$. Since X and Y are uniformly chosen, then their joint PDF is

$$f(x,y) = \begin{cases} 1 & 0 < x, y < 1 \\ 0 & \text{otherwise} \end{cases}.$$

Hence,

$$\begin{aligned} E[|X-Y|] &= \int_0^1 \int_0^1 |x-y| \cdot \, dx\, dy \\ &= \iint_{x \ge y} (x-y)\, dx\, dy + \iint_{y > x} (y-x)\, dx\, dy \\ &= \int_0^1 \int_y^1 (x-y)\, dx\, dy + \int_0^1 \int_0^y (y-x)\, dx\, dy \\ &= \frac{1}{3}. \end{aligned}$$

Exercise 3 Gasoline is stocked in a bulk tank each week at a particular gas station. Let X be the random variable that represents the proportion of the tank's capacity stocked in a given week, and Y be the random variable that represents the proportion of the tank's capacity sold in the same week. Note that the gas station cannot sell more than what was stocked in a given week, which implies that the value of Y cannot exceed the value of X. A possible joint pdf of X and Y is determined as follows

$$f(x,y) = \begin{cases} 3x, & \text{if } 0 \le y \le x \le 1 \\ 0, & \text{otherwise .} \end{cases}$$

1- Find the joint CDF of X and Y at the point $(x,y) = (1/2, 1/3)$.

2- What is the probability that the amount of gas sold is less than half the amount that is stocked in a given week.

Solution

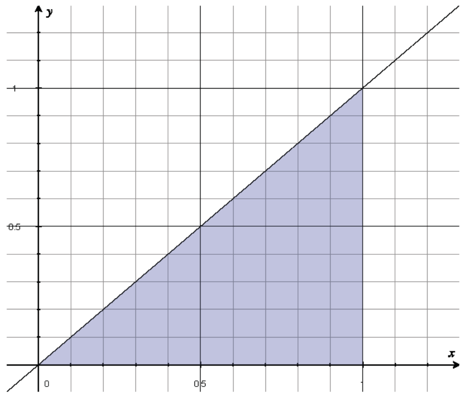

FIGURE 1: Region over which joint pdf $f(x, y)$ is nonzero.

1-

$$F\left(\frac{1}{2}, \frac{1}{3}\right) = P\left(X \leq \frac{1}{2}, Y \leq \frac{1}{3}\right) = \int_0^{1/3} \int_y^{0.5} 3x dx dy$$
$$= \int_0^{1/3} \left(\frac{3}{2}x^2 \Big|_y^{0.5} \right) dy = \int_0^{1/3} \left(\frac{3}{8} - \frac{3}{2}y^2 \right) dy$$
$$= \frac{3}{8}y - \frac{1}{2}y^3 \Big|_0^{1/3} \approx 0.1065$$

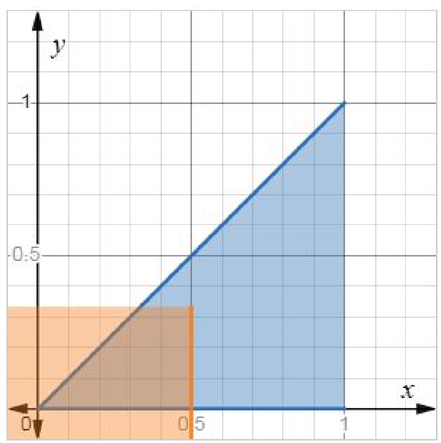

FIGURE 2: Intersection of $\{(x, y) \mid x \leq 1/2, y \leq 1/3\}$ with the region over which joint pdf $f(x, y)$ is nonzero.

2- Finding the probability that the amount of gas sold is less than half the amount that is stocked in a given week means that to compute $P(Y < 0.5X)$. Figure 3 illustrates the intersection of $\{(x, y) \mid y < 0.5x\}$ with the feasible PDF region over which joint pdf $f(x, y)$ is nonzero.

$$P(Y < 0.5X) = \int_0^1 \int_0^{0.5x} 3x dy dx$$
$$= \int_0^1 \left(3xy|_0^{0.5x} \right) dx$$
$$= \int_0^1 \left(\frac{3}{2}x^2 - 0 \right) dx = \frac{1}{2}x^3 \Big|_0^1$$
$$= \frac{1}{2}$$

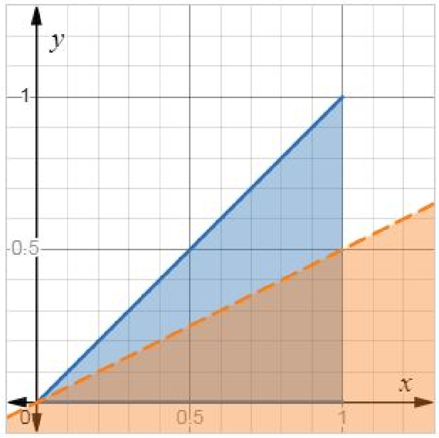

FIGURE 3: Intersection of $\{(x, y) \mid y < 0.5x\}$ with the feasible PDF region over which joint pdf $f(x, y)$ is nonzero.

Exercise 4 Let X and Y two random variables with joint PMF given in Table1 below.
1- Find $P(X \leq 2, Y \leq 4)$.
2- Find the marginal PMF's of X and Y.
3- Find $P(Y = 2|X = 1)$.
4- Are X and Y are independent.

Solution

$f(x,y)$	Y		
X	2	4	5
1	1/12	1/24	1/24
2	1/6	1/12	1/8
3	1/4	1/8	1/12

TABLE 1: Joint PMF of X and Y

1-

$$\begin{aligned} P(X \leq 2, Y \leq 4) &= P_{XY}(1,2) + P_{XY}(1,4) + P_{XY}(2,2) + P_{XY}(2,4) \\ &= \frac{1}{12} + \frac{1}{24} + \frac{1}{6} + \frac{1}{12} = \frac{3}{8}. \end{aligned}$$

2- We have $S_X = \{1, 2, 3\}$ and $S_Y = \{2, 4, 5\}$, then

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{11}{24} & x = 3 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(y) = \begin{cases} \frac{1}{2} & y = 2 \\ \frac{1}{4} & y = 4 \\ \frac{1}{4} & y = 5 \\ 0 & \text{otherwise} \end{cases}$$

3- By the conditional probability formula, we have

$$\begin{aligned} P(Y=2|X=1) &= \frac{P(X=1,Y=2)}{P(X=1)} \\ &= \frac{P_{XY}(1,2)}{P_X(1)} \\ &= \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2}. \end{aligned}$$

4- Since we have

$$P(X=2,Y=2) = \frac{1}{6} \neq P(X=2)P(Y=2) = \frac{3}{16}.$$

it results that X and Y are not independent.

Exercise 5 Let X and Y two random variables with joint PMF given in Table2 below.
1- Find $P(X=0, Y \leq 1)$.
2- Find the marginal PMF's of X and Y.
3- Find $P(Y=1|X=0)$.
4- Are X and Y are independent.

$f(x,y)$	Y		
X	0	1	2
0	1/6	1/4	1/8
1	1/8	1/6	1/6

TABLE 2: Joint PMF of X and Y

Solution
1-

$$P(X=0, Y \leq 1) = P_{XY}(0,0) + P_{XY}(0,1) = \frac{1}{6} + \frac{1}{4} = \frac{5}{12}.$$

2- We have $S_X = \{0,1\}$ and $S_Y = \{0,1,2\}$. To find for example $P_X(0)$, we can write

$$\begin{aligned} P_X(0) &= P_{XY}(0,0) + P_{XY}(0,1) + P_{XY}(0,2) \\ &= \frac{1}{6} + \frac{1}{4} + \frac{1}{8} \\ &= \frac{13}{24}. \end{aligned}$$

We use the same technique to compute the other probabilities, we obtain

$$P_X(x) = \begin{cases} \frac{13}{24} & x = 0 \\ \frac{11}{24} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(y) = \begin{cases} \frac{7}{24} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{7}{24} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

3-

$$\begin{aligned}P(Y=1|X=0) &= \frac{P(X=0,Y=1)}{P(X=0)}\\ &= \frac{P_{XY}(0,1)}{P_X(0)}\\ &= \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}.\end{aligned}$$

4- Since we have

$$P(Y=1|X=0) = \frac{6}{13} \neq P(Y=1) = \frac{5}{12}.$$

it results that X and Y are not independent.

Exercise 6 Let X and Y be two continuous random variables having the following joint PDF

$$f_{XY}(x,y) = \begin{cases} x+cy^2 & 0 \leq x,y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1- Find the constant c
2- Find $P\left(0 \leq X, Y \leq \frac{1}{2}\right)$.
3- Find the marginal PMF's of X and Y
4- Find the joint CDF for X and Y
Solution
1- Since we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$$

Thus,

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy\\ &= \int_0^1\int_0^1 x+cy^2dxdy\\ &= \int_0^1 \left[\frac{1}{2}x^2+cy^2x\right]_{x=0}^{x=1} dy\\ &= \int_0^1 \frac{1}{2}+cy^2dy\\ &= \left[\frac{1}{2}y+\frac{1}{3}cy^3\right]_{y=0}^{y=1}\\ &= \frac{1}{2}+\frac{1}{3}c\end{aligned}$$

Therefore, we obtain $c = \frac{3}{2}$.
2-

$$\begin{aligned}P\left(0 \leq X, Y \leq \frac{1}{2}\right) &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} \left(x+\frac{3}{2}y^2\right) dxdy\\ &= \int_0^{\frac{1}{2}} \left[\frac{1}{2}x^2+\frac{3}{2}y^2x\right]_0^{\frac{1}{2}} dy\\ &= \int_0^{\frac{1}{2}} \left(\frac{1}{8}+\frac{3}{4}y^2\right) dy\\ &= \frac{3}{32}\end{aligned}$$

3- The marginal PMF's of X and Y.
For $0 \leq x \leq 1$, we have

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^1 \left(x + \frac{3}{2}y^2\right) dy \\ &= \left[xy + \frac{1}{2}y^3\right]_0^1 \\ &= x + \frac{1}{2} \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} x + \frac{1}{2} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, for $0 \leq y \leq 1$, we have

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_0^1 \left(x + \frac{3}{2}y^2\right) dx \\ &= \left[\frac{1}{2}x^2 + \frac{3}{2}y^2 x\right]_0^1 \\ &= \frac{3}{2}y^2 + \frac{1}{2} \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{3}{2}y^2 + \frac{1}{2} & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

4- To find the joint CDF for $x > 0$ and $y > 0$, we need to integrate the joint PDF :

$$\begin{aligned} F_{XY}(x,y) &= \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(u,v)dudv \\ &= \int_0^y \int_0^x f_{XY}(u,v)dudv \\ &= \int_0^{\min(y,1)} \int_0^{\min(x,1)} \left(u + \frac{3}{2}v^2\right) dudv. \end{aligned}$$

For $0 \leq x, y \leq 1$, we obtain

$$\begin{aligned} F_{XY}(x,y) &= \int_0^y \int_0^x \left(u + \frac{3}{2}v^2\right) dudv \\ &= \int_0^y \left[\frac{1}{2}u^2 + \frac{3}{2}v^2 u\right]_0^x dv \\ &= \int_0^y \left(\frac{1}{2}x^2 + \frac{3}{2}xv^2\right) dv \\ &= \frac{1}{2}x^2 y + \frac{1}{2}xy^3. \end{aligned}$$

For $0 \leq x \leq 1$ and $y \geq 1$, we use the fact that F_{XY} is continuous to obtain

$$\begin{aligned} F_{XY}(x,y) &= F_{XY}(x,1) \\ &= \frac{1}{2}x^2 + \frac{1}{2}x. \end{aligned}$$

Similarly, for $0 \leq y \leq 1$ and $x \geq 1$, we obtain

$$\begin{aligned} F_{XY}(x, y) &= F_{XY}(1, y) \\ &= \frac{1}{2}y + \frac{1}{2}y^3. \end{aligned}$$

Exercise 7 Let X and Y be two continuous random variables having the following joint PDF

$$f_{XY}(x, y) = \begin{cases} cx^2 y & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1- Determinate $S_{X,Y}$ and show it in the plane.
2- Determinate the constant c.
3- Determinate the marginal PDFs, $f_X(x)$ and $f_Y(y)$.
4- Determinate $P\left(Y \leq \frac{X}{2}\right)$.
5- Determinate $P\left(Y \leq \frac{X}{4} \middle| Y \leq \frac{X}{2}\right)$.

Solution

We have

$$S_{X,Y} = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq x \leq 1\right\}$$

Figure 4 shows S_{XY}.

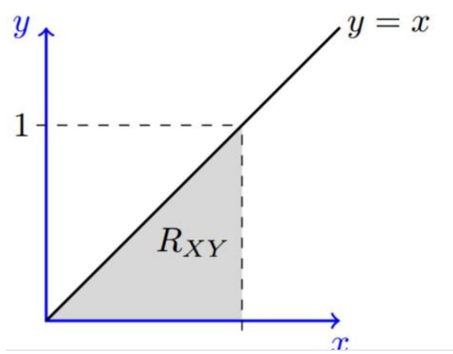

FIGURE 4: Region over which joint pdf $f(x, y)$ is nonzero.

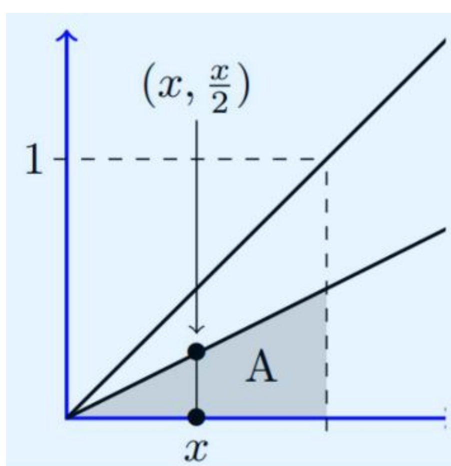

FIGURE 5: Intersection of $\{(x, y) \mid y < 0.5x\}$ with the feasible PDF region over which joint pdf $f(x, y)$ is nonzero.

2- To find the constant c, we can write

$$\begin{aligned}1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_0^1\int_0^x cx^2 ydydx \\ &= \int_0^1 \frac{c}{2}x^4 dx \\ &= \frac{c}{10}\end{aligned}$$

Thus, $c = 10$

3- The marginal PMF's of X and Y. For $0 \leq x \leq 1$, we have

$$\begin{aligned}f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^x 10x^2 ydy \\ &= 5x^4\end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ 0 & \text{otherwise}\end{cases}$$

Similarly, for $0 \leq y \leq 1$, we set

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_y^1 10x^2 ydx \\ &= \frac{10}{3}y\left(1-y^3\right)\end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{10}{3}y\left(1-y^3\right) & 0 \leq y \leq 1 \\ 0 & \text{otherwise}\end{cases}$$

4-

$$\begin{aligned}P\left(Y \leq \frac{X}{2}\right) &= \int_{-\infty}^{\infty}\int_0^{\frac{x}{2}} f_{XY}(x,y)dydx \\ &= \int_0^1\int_0^{\frac{x}{2}} 10x^2 ydydx \\ &= \int_0^1 \frac{5}{4}x^4 dx \\ &= \frac{1}{4}\end{aligned}$$

5-

$$\begin{aligned}
P\left(Y \leq \frac{X}{4} \middle| Y \leq \frac{X}{2}\right) &= \frac{P\left(Y \leq \frac{X}{4}, Y \leq \frac{X}{2}\right)}{P\left(Y \leq \frac{X}{2}\right)} \\
&= 4P\left(Y \leq \frac{X}{4}\right) \\
&= 4\int_0^1 \int_0^{\frac{x}{4}} 10x^2 y dy dx \\
&= 4\int_0^1 \frac{5}{16} x^4 dx \\
&= \frac{1}{4}
\end{aligned}$$