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Chapter 4 : Conditional probabilities and independence : Conditional distribution of a discrete random variable

1. Conditional distribution of a discrete random variable

Recall that for any two events A and B such that $P(B) > 0$, the conditional probability is defined as follows

$$P(E|F) = \frac{P(E \cap F)}{P(F)}.$$

We use this same concept for events to define conditional probabilities for random variables.

Définition 1. Let X and Y be discrete random variables and $p(x, y)$ is the joint probability mass function. The conditional probability mass function of X, given that $Y = y$ is defined as

$$\begin{aligned} p_{X|Y}(x|y) &= P(X = x|Y = y) \\ &= \begin{cases} \frac{p_{X,Y}(x,y)}{p_Y(y)} & \text{if } p_Y(y) > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

where $p_Y(y)$ is the marginal PMF of Y for any x and y such that $p_Y(y) > 0$

Définition 2. (**Conditional CDF of discrete random variable**) Let X and Y be discrete random variables, the conditional cumulative distribution function of X given $Y = y$ is

$$F_{X|Y}(x|y) = P(X \leq x|Y = y) = \sum_{x' \leq x} p_{X|Y}(x'|y).$$

1.1. Properties of the conditional probability mass function

-The conditional probability mass function for X, given $Y = y$, for a fixed y, is a PMF satisfying the following :

$$0 \leq p_{X|Y}(x|y) \leq 1 \qquad \text{and} \qquad \sum_x p_{X|Y}(x|y) = 1,$$

indeed,

$$\begin{aligned} \sum_x p_{X|Y}(x|y) &= \sum_x P(X = x|Y = y) \\ &= \sum_x \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \frac{1}{p_Y(y)} \sum_x p_{X,Y}(x,y) \\ &= \frac{1}{p_Y(y)} p_Y(y) \\ &= 1. \end{aligned}$$

- Typically, the conditional distribution of X given Y is different from the conditional distribution of Y given X, i.e.

$$p_{X|Y}(x|y) \neq p_{Y|X}(y|x).$$

- If X and Y are independent, discrete random variables, then :

$$\begin{aligned} p_{X|Y}(x|y) &= p_X(x) \\ p_{Y|X}(y|x) &= p_Y(y) \end{aligned}$$

Exemple 1. We have investigated the correlation between hair and eye color among randomly selected Saint Mary's students. Here is the joint probability mass function that we have obtained, along with the marginal PMF's provided in the margins :

$p(x,y)$	Hair Color (X)				
Eye Color (Y)	blonde (1)	red (2)	brown (3)	black (4)	$p_Y(y)$
blue (1)	0.12	0.05	0.12	0.01	0.30
green (2)	0.12	0.07	0.09	0	0.28
brown (3)	0.16	0.07	0.16	0.03	0.42
$p_X(x)$	0.40	0.19	0.37	0.04	1.00

1- Find the portion of SMC students with blue eyes having red hair.
2- Find portion of SMC students with blonde hair having green eyes.
solution
1 - First, let's determine $p_{X|Y}(2 \mid 1)$:

$$p_{Y|X}(1 \mid 2) = \frac{p_{Y,X}(1,2)}{p_Y(2)} = \frac{0.05}{0.19} = \frac{5}{19} \approx 0.167$$

There are approximately 16.7% of students with blue eyes having red hair.
2 - Now, let's reverse the order of X and Y, and find $p_{Y|X}(2 \mid 1)$:

$$p_{X|Y}(1 \mid 2) = \frac{p_{X,Y}(1,2)}{p_Y(2)} = \frac{0.12}{0.28} = \frac{3}{7}$$

There are approximately 30% of SMC students with blonde hair having green eyes.

Exemple 2. Suppose the joint PMF of X and Y is given by the following probability table.

X/Y	y = 0	y = 1	y = 2	y = 3
x = 0	0	$\frac{1}{42}$	$\frac{2}{42}$	$\frac{3}{42}$
x = 1	$\frac{2}{42}$	$\frac{3}{42}$	$\frac{4}{42}$	$\frac{5}{42}$
x = 2	$\frac{4}{42}$	$\frac{5}{42}$	$\frac{6}{42}$	$\frac{7}{42}$

- Determine the conditional PMF of Y given $X = 1$.

Solution
We have

$$P_{Y|X}(y \mid x = 1) = \frac{p_{X,Y}(x = 1, y)}{p_X(x = 1)} = \frac{p_{X,Y}(x = 1, y)}{14/42}$$

Hence, the conditional PMF of Y given $X = 1$ is as follows

$$P_{Y|X}(y \mid x = 1) = \begin{cases} \frac{2/42}{14/42} = \frac{2}{14}, & \text{if } y = 0 \\ \frac{3/42}{14/42} = \frac{3}{14}, & \text{if } y = 1 \\ \frac{4/42}{14/42} = \frac{4}{14}, & \text{if } y = 2 \\ \frac{5/42}{14/42} = \frac{5}{14}, & \text{if } y = 3 \end{cases}$$