Ferhat Abbas-Sétif 1 University
Faculty of Sciences
Department Informatics

Chapter 4 : Conditional probabilities and independence : Conditional distributions of continuous random variables

1. Conditional distributions of continuous random variables

The following definition gives the formulas for conditional distributions of continuous random variables by simply replacing sums with integrals and PMF's with PDF's

Définition 1. (Conditional PDF) If X and Y are continuous random variables with joint probability density function (PDF) given by $f(x,y)$, is the joint probability mass function. The conditional probability density function of X, given that $Y=y$, is defined by

$$f_{X|Y}(x|y) = \begin{cases} \frac{f_{X,Y}(x,y)}{f_Y(y)}, & \text{if } f_Y(y) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Définition 2. (Conditional CDF of continuous random variable) If X and Y are continuous random variables, the conditional cumulative distribution function of X given $Y=y$ is

$$F_{X|Y}(x|y) = P(X \leq x|Y=y) = \int_{-\infty}^{x} f_{X|Y}(t|y)dt.$$

1.1. Properties of conditional probability density function

1. The conditional PDF for X, given $Y=y$, for a fixed y, is a PDF satisfying the following :

$$0 \leq f_{X|Y}(x \mid y) \quad \text{and} \quad \int_{\mathbb{R}} f_{X|Y}(x \mid y)dx = 1$$

2. Typically, the conditional distribution of X given Y does not equal the conditional distribution of Y given X, i.e.,

$$f_{X|Y}(x \mid y) \neq f_{Y|X}(y \mid x)$$

3. If X and Y are independent, continuous random variables, then the following are true :

$$f_{X|Y}(x \mid y) = f_X(x)$$
$$f_{Y|X}(y \mid x) = f_Y(y)$$

2. Conditional expectation and variance

Définition 3. Let X and Y discrete random variables, then the conditional expected value of $\boldsymbol{X}$, given $\boldsymbol{Y}=\boldsymbol{y}$, is given by

$$\mu_{X|Y=y} = \mathrm{E}[X \mid Y=y] = \sum_x x p_{X|Y}(x \mid y)$$

and the conditional variance of $\boldsymbol{X}$, given $\boldsymbol{Y}=\boldsymbol{y}$, is given by

$$\begin{aligned}\sigma^2_{X|Y=y} = \mathrm{Var}(X \mid Y=y) = \mathrm{E}\left[\left(X-\mu_{X|Y=y}\right)^2 \mid Y=y\right] &= \sum_{x\in S_X} \left(x-\mu_{X|Y=y}\right)^2 p_{X|Y}(x \mid y) \\ &= \mathrm{E}\left[X^2 \mid Y=y\right] - \mu^2_{X|Y=y} = \left(\sum_{x\in S_X} x^2 p_{X|Y}(x \mid y)\right) - \mu^2_{X|Y=y}\end{aligned}.$$

Similarly, if X and Y are continuous random variables with joint PDF given by $f(x, y)$, then the conditional expected value of $\boldsymbol{X}$, given $\boldsymbol{Y} = \boldsymbol{y}$, is

$$\mathrm{E}[X \mid Y = y] = \int_{\mathbb{R}} x f_{X|Y}(x \mid y) dx$$

and the conditional variance of $\boldsymbol{X}$, given $\boldsymbol{Y} = \boldsymbol{y}$, is

$$\begin{aligned}\operatorname{Var}(X \mid Y = y) &= \mathrm{E}\left[X^2 \mid Y = y\right] - (\mathrm{E}[X \mid Y = y])^2 \\ &= \int_{\mathbb{R}} x^2 f_{X|Y}(x \mid y) dx - \left(\int_{\mathbb{R}} x f_{X|Y}(x \mid y) dx\right)^2.\end{aligned}$$