Chapter 1

Sets, relations and applications

Notations :

$:=$: means "define";	$\in$: "belongs to" $a \in S$ means that "a is an element in S.";
$\exists$: means "there exists";	$\exists!$: means "there exists a unique";
$\forall$: "for all";	$\notin$: "does not belong to";
$\subset$: "contained in";	$\subseteq$: "content or equal to";
$\not\subset$: ‹‹is not contained in››;	$\forall$: means "for all";
$\Rightarrow$: means "implies";	$\Longleftrightarrow$; means "if and only if."

You already know a some famous sets :

· Set of Natural Numbers is denoted by $\mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, 3, ...\}$).

· Set of Integers is denoted by $\mathbb{Z}$ ($\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$).

· Set of Rational Numbers is denoted by $\mathbb{Q}$ ($\mathbb{Q} = \{\frac{p}{q} \,/\, p \in \mathbb{Z}, p \in (\mathbb{N}^*)\}$).

· Set of Real Numbers is denoted by $\mathbb{R}$ for example : 1, $\sqrt{3}$, Π, $\ln 3$,...

· The set of complex numbers $\mathbb{C}$ for example : $1 + 3i$,...

We will try to see the properties of sets, without focusing on a particular example. You will quickly realize that what is at least as important as sets are the relations between sets : this will be the notion of application (or function) between two sets.

1.1 Sets

1.1.1 Define sets

► A set is a collection of objects that verify certain properties. An object which satisfies the needed rules is called element of the set. If the set is denoted by A and x is an element of A we say x belongs to A and we write $x \in A$

Example 1.1.1 *(i)* $A = \{0, 1\}$. *This means that the set* $\mathcal{A}$ *consists of two elements,* 0 *and* 1.

(ii) $B = \{x \in \mathbb{R} : -3 < x \leqslant 2\} =]-3, 2]$.

(iii) $C = \{0, \{1\}, \{0, 1\}\}$. *The set* C *contains three elements: the number* 0*; the set* $\{1\}$ *containing one element, namely the number* 1*; and the set containing two elements, the numbers* 0 *and* 1.

► The order in which the elements are listed is not important. Like this $\{0, 1\} = \{1, 0\}$. An element may occur more than once. So $\{1, 2, 1\} = \{1, 2\}$. But $\{1, 2, \{1\}\} \neq \{1, 2\}$!

A set can be also specified by an elementhood test.

1.1.2 cardinality of a finite set

If a set A contains a finite number of elements it is said to be finite, otherwise it is said to be infinite. If A is finite and it contains $n \in \mathbb{N}$ elements, then n is called the cardinality of A we write $card\ A = n$ or $|A| = n$. If $n = 0$ the set A is called an empty set and is denoted by $\emptyset$ and we have $card\ A = 0$.

Definition 1.1.2 *The empty set is the set which contains no elements, and is denoted by* $\emptyset$.

In the previous example B is infinite set, $|A| = 2$ and $|C| = 3$.

1.1.3 Operations on sets

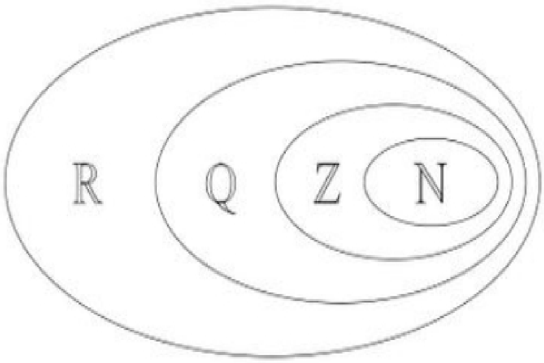
Now we introduce operations on sets. The main operations are: Inclusion, union, intersection, difference and symmetric difference.

Definition 1.1.3 ***1.*** *A set* A *is a subset of* B, $A \subset B$, *if every element of* A *is in* B. *Given* $A \subset B$, *if* $a \in A \Longrightarrow a \in B$.

2. *Two sets A and B are equal,* $A = B$*, if* $A \subset B$ *and* $B \subset A$.
3. *A set A is a proper subset of B,* $A \subsetneq B$ *if* $A \subset B$ *and*$A \neq B$

Thus, one way to show that two sets, A and B, coincide is to show that each element in A is contained in B and vice-versa.

Example 1.1.4 *We have* $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

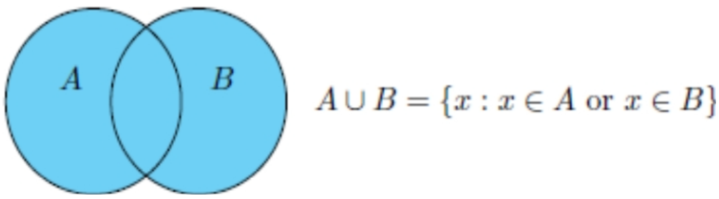


Definition 1.1.5 *The union of sets A and B is the set containing the elements of A and the elements of B, and no other elements.*

Notation 1 *We denote the union of A and B by* $A \cup B$.

Note: existence of the union for arbitrary A and B is accepted as an axiom. For arbitrary x and arbitrary A and B the following proposition is true.

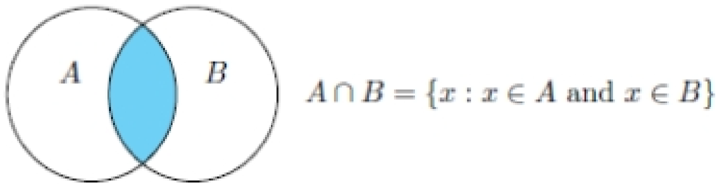
$$(x \in A \cup B) \Leftrightarrow (x \in A) \vee (x \in B).$$



Definition 1.1.6 *The intersection of sets A and B is the set containing the elements which are elements of both A and B, and no other elements.*

We denote the intersection of A and B by $A \cap B$. Thus for arbitrary x and arbitrary A and B the following proposition is true.

$$(x \in A \cap B) \Leftrightarrow (x \in A) \wedge (x \in B).$$

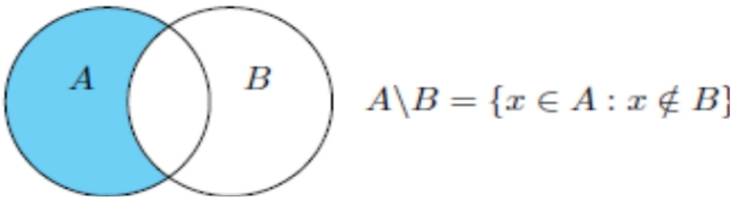


Note: When $A \cap B = \emptyset$, then A and B are said to be disjoint.

Definition 1.1.7 *The difference of sets A and B is the set containing the elements of A which do not belong to B.*

We use the notation $A - B$ for the difference or the complement of B with respect to A $(A \backslash B)$. The following is true for arbitrary x and arbitrary A and B :

$$(x \in A - B) \Leftrightarrow [(x \in A) \wedge (\not{x} \in B)].$$

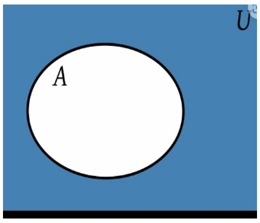


Definition 1.1.8 *The symmetric difference of the sets A and B is defined by*

$$A \triangle B = (A - B) \cup (B - A).$$

Definition 1.1.9 *Suppose that $A \subset U$. The complement of the set A in U denoted by A^c, $\complement_U(A)$ or $\bar{A}$, is the set of all elements of U that are not in A. That is $A^c = \{x \in U,$ $x \notin A\}$.*

Let us illustrate these operations with a simple example.



Example 1.1.10 *Let* $U = \mathbb{N}$, $A = \{0, 1, 2, 3, 4, 5\}$ *and* $B = \{1, 3, 5, 7, 9\}$*. Then*

$$\begin{array}{lcl} A \cup B &=& \{0, 1, 2, 3, 4, 5, 7, 9\}. \\ A \cap B &=& \{1, 3, 5\}. \\ A - B &=& \{0, 2, 4\}. \\ B - A &=& \{7, 9\}. \\ A \triangle B &=& \{0, 2, 4, 7, 9\}. \\ A^c &=& \{k : k \in \mathbb{N} \text{ and } k \geq 6\} = \{6, 7, ...\} \end{array}$$

Note that

$$A \cup B = (A \cap B) \cup (A \triangle B)$$

1.1.4 Laws for operations on sets

Let A, B be subsets of an universal set U

Idempotent Laws	(a) $A \cup A = A$	(b) $A \cap A = A$
Associative Laws	(a) $(A \cup B) \cup C = A \cup (B \cup C)$	(b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	(a) $A \cup B = B \cup A$	(b) $A \cap B = B \cap A$
Distributive Laws	(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(b) $A \cap (B \cup C) =(A \cap B) \cup (A \cap C)$
De Morgan's Laws	(a) $(A \cup B)^c=A^c\cap B^c$	(b) $(A \cap B)^c=A^c\cup B^c$
Identity Laws	(a) $A \cup \emptyset = A$ (b) $A \cup U = U$	(c) $A \cap U =A$ (d) $A \cap \emptyset = \emptyset$
Complement Laws	(a) $A \cup A^c= U$ (b) $A \cap A^c= \emptyset$	(c) $U^c= \emptyset$ (d) $\emptyset^c = U$
Involution Law	(a) $(A^c)^c = A$	

A few demonstrations * $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$?

$$
\begin{aligned}
x \in A \cap (B \cup C) &\Leftrightarrow (x \in A \text{ and } x \in (B \cup C)) \\
&\Leftrightarrow (x \in A \text{ and } (x \in B \text{ or } x \in C)) \\
&\Leftrightarrow ((x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)) \\
&\Leftrightarrow ((x \in A \cap B) \text{ or } (x \in A \cap C)) \\
&\Leftrightarrow x \in (A \cap B) \cup (A \cap C).
\end{aligned}
$$

Then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?

$$
\begin{aligned}
(x \in A \cup (B \cap C)) &\Leftrightarrow (x \in A \text{ or } x \in B \cap C) \\
&\Leftrightarrow (x \in A \text{ or} (x \in B \text{ and } x \in C)) \\
&\Leftrightarrow ((x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C)) \\
&\Leftrightarrow (x \in A \cup B \text{ and } x \in A \cup C) \\
&\Leftrightarrow x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Then $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

* $\complement_U(A \cap B) = \complement_U(A) \cup \complement_U(B)$ and $\complement_U(A \cup B) = \complement_U(A) \cap \complement_U(B)$?

- $\left(x \in \complement_U(A \cap B)\right) \Leftrightarrow (x \notin A \cap B) \Leftrightarrow (x \notin A \text{ or } x \notin B)$
$\Leftrightarrow \left(x \in \complement_U(A) \text{ or } x \in \complement_U(B)\ \right)$
$\Leftrightarrow \left(x \in \complement_U(A)\ \cup \complement_U(B)\ \right)$

Therefore $\complement_U(A \cap B) = \complement_U(A) \cup \complement_U(B)$

- $\left(x \in \complement_U(A \cup B)\right) \Leftrightarrow (x \notin A \cup B) \Leftrightarrow (x \notin A \text{ and } x \notin B)$
$\Leftrightarrow \left(x \in \complement_U(A) \text{ and } x \in \complement_U(B)\right)$
$\Leftrightarrow \left(x \in \complement_U(A)\ \cap \complement_U(B)\right).$

Therefore $\complement_U(A \cup B) = \complement_U(A) \cap \complement_U(B)$.

* $\complement_U\left(\complement_U(A)\right) = A$?

$\left(x \in \complement_U\left(\complement_U(A)\right)\right) \Leftrightarrow \left(x \notin \left(\complement_U(A)\right)\right) \Leftrightarrow \overline{\left(x \in \complement_U(A)\right)}$
$\Leftrightarrow \overline{(x \notin A)}$
$\Leftrightarrow x \in A.$

1.1.5 Set of parts.

Definition 1.1.11 *Let E be a set, we form a set called the set of parts of E, denoted $P(E)$ which is characterized by the following relation $P(E) = \{A : A \subset E\}$*

Example 1.1.12 *Let $E = \{0, 1, 2\}$. Then*

$$P(E) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}.$$

Remark 1.1.13 *If card $(A) = n$ then card $(P(A)) = 2^n$..*

Example 1.1.14 - *If $E = \{a, b\}$,then*

$$P(E) = \{\emptyset, \{a\}, \{b\}, \{a,b\}\},$$

as card $(E) = 2$, then card $(P(E)) = 2^2 = 4$

-*If $E = \{a\}$, then $P(A) = \{\emptyset, \{a\}\}$.*

1.1.6 Cartesian product

Definition 1.1.15 . *Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) in which $a \in A$ and $b \in B$, i.e.*

$A \times B = \{(a, b) : (a \in A) \wedge (b \in B)\}$.

Thus

$$(p \in A \times B) \Leftrightarrow \{(\exists a \in A)(\exists b \in B)[p = (a, b)]\}.$$

Example 1.1.16 *(i) If* $A = \{red, green\}$ *and* $B = \{1, 2, 3\}$ *then*

$$A \times B = \{(red, 1), (red, 2), (red, 3), (green, 1), (green, 2), (green, 3)\}.$$

(ii) $\mathbb{Z} \times \mathbb{Z} = \{(x, y) |\ x$ *and* y *are integers*$\}$. *This is the set of integer coordinates points in the* $x, y-$*plane. The notation* $\mathbb{Z}^2$ *is usually used for this set.*

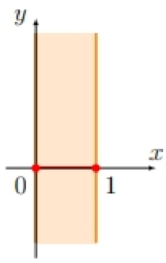
Example 1.1.17 *If* $E = \{1, 2\}$ *and* $F = \{3, 5\}$, *then*

$$E \times F = \{(1, 3), (1, 5), (2, 3), (2, 5)\}$$
$$F \times\ E = \{(3, 1), (3, 2), (5, 1), (5, 2)\}$$

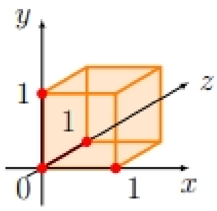
$E \times F \neq F \times\ E$

Example 1.1.18 *1)* $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$

2) $[0, 1] \times \mathbb{R} = \{(x, y) : 0 \leq x \leq 1,\ y \in \mathbb{R}\}$



Example 1.1.19 $[0, 1] \times [0, 1] \times [0, 1] = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$



Notation 2 *Let* E^2 *be the Cartesian square of* E. *More generally, we define the Cartesian product of* n *sets* $E_1,\ E_2,\ ...,E_n$ *by*

$$E_1 \times E_2 \times ... \times E_n = \{(x_1, x_2, ..., x_n) : xi \in E_i\ ,\ \textit{for}\ i = 1, ..., n\}.$$

Example 1.1.20 *If* $E = \{1, 2\}$, *then*

$E^2 = E \times E = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

$E^3 = E \times E \times E = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 2, 1), (2, 1, 2), (2, 2, 2)\}$

Proposition 1.1.21 *let E and F be two finite sets.Then*
$card\ (E \times F) = card\ (E) \times card\ (F)$

The following theorem provides some basic properties of the Cartesian product.

theorem 1.1.22 *LetA, B, C, D be sets. Then*
a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$,
b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$,
c) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$,
d) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$,
e) *If A or B are empty sets (($A = \emptyset$ and $B \neq \emptyset$) or ($B = \emptyset$ and $A \neq \emptyset$) or ($A = \emptyset$ and $B = \emptyset$)), then $A \times B = \emptyset$.*

proof. (**a**) $(\Rightarrow)$
Let $p \in A \times (B \cap C)$. Then

$$(\exists a \in A)(\exists x \in B \cap C)[p = (a, x)]$$

In particular,

$$(\exists a \in A)(\exists x \in B)[p = (a, x)] \text{ and } (\exists a \in A)(\exists x \in C)[p = (a, x)]$$

So $p \in (A \times B) \cap (A \times C)$.
(**a**) $(\Leftarrow)$
Let $p \in (A \times B) \cap (A \times C)$. Then

$$p \in (A \times B) \text{ and } p \in (A \times C).$$

So

$$(\exists a \in A)(\exists b \in B)[p = (a, b)] \text{ and } (\exists \grave{a} \in A)(\exists c \in C)[p = (\grave{a}, c)]$$

But then $(a, b) = p = (\grave{a}, c)$ and hence $a = \grave{a}$ and $b = c$. Thus $p = (a, x)$ for some $a \in A$ and $x \in B \cap C$, i.e.$p \in A \times (B \cap C)$. This proves (**a**) **.** ■

The proof of (**b**), (**c**), (**d**) and (**e**) are left as exercises.

1.2 Relations, Equivalence Relation

1.2.1 Relations

Definition 1.2.1 *We call the relation $\mathcal{R}$ from E to F any part of the Cartesian product $E \times F$. The domain of $\mathcal{R}$ is the set*

$$D(\mathcal{R}) = \{x \in E : \exists y \in F\ [(x, y) \in \mathcal{R}]\}.$$

The range of $\mathcal{R}$ *is the set*

$$Ran(\mathcal{R}) = \{y \in F : \exists x \in E[(x, y) \in \mathcal{R}]\}.$$

If $E = F$*, we say that* $\mathcal{R}$ *is a binary relation on* E*.*
The inverse of $\mathcal{R}$ *is the relation* $\mathcal{R}^{-1}$ *from* F *to* E *defined as follows*

$$\mathcal{R}^{-1} = \{(y, x) \in F \times E : (x, y) \in \mathcal{R}\}.$$

The graph of this relation is:

$$G_R = \{(x, y) \in E \times F : x \ \mathcal{R} \ y \ \}$$

Example 1.2.2 ***(i)*** *Let* $A = \{1, 2, 3\}$*,* $B = \{3, 4, 5\}$*. The set* $\mathcal{R} = \{(1, 3), (1, 5), (3, 3)\}$ *is a relation from* A *to* B *since* $\mathcal{R} \sqsubseteq A \times B$*.*

(ii) $G = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x > y\}$ *is a relation from* $\mathbb{Z}$ *to* $\mathbb{Z}$*.*

Example 1.2.3 *Let* $A = \{1, 2, 3, 4, 5, 6\}$ *a set and the relation* $\mathcal{R}$ *defined by*

$$x\mathcal{R}y \Leftrightarrow x \text{ divide } y \text{ (in } \mathbb{Z})$$

$$\begin{aligned} G_R &= \{(x, y) \in A \times A, \quad x \text{ divide } y \ \} \\ &= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}. \end{aligned}$$

Definition 1.2.4 *Let* $\mathcal{R}$ *be a binary relation over a set* E*. For all* $x, y, z \in E$*, we say that* $\mathcal{R}$ *is*

(1) Reflexives*, if each element is related to itself, i.e*

$$.x\mathcal{R}x, \forall x \in E$$

(2) Symmetric*, if for all* $x, y \in E$*, if* x *is related to* y *then* y *is related to* x*, i.e.*$x\mathcal{R}y \Rightarrow y\mathcal{R}x, \forall x, y \in E$*.*

(3) Transitive*, if for all* $x, y, z \in E$*, if* x *is in relation to* y *and* y *in relation to* z *then* x *is in relation to* z*, i.e.* $(x\mathcal{R}y \text{ and } y\mathcal{R}z) \Rightarrow x\mathcal{R}z, \forall x, y, z \in E$*.*

(4) Anti-symmetric*, if two elements are related to each other, then they are equal, i.e.*

$$(x\mathcal{R}y \text{ and } y\mathcal{R}x) \Rightarrow x = y, \forall x, y \in E.$$

A particularly important class of relations are equivalence relations.

1.2.2 Equivalence Relation

Definition 1.2.5 *A relation $\mathcal{R}$ on E is called equivalence relation if it is reflexive, symmetric and transitive.*

Example 1.2.6 ***(i)*** *Let E be a set of students. A relation on $E \times E$: "to be friends". It is reflexive (I presume that everyone is a friend to himself / herself). It is symmetric. But it's not transitive.*

(ii) *Let $E = Z$,$a \in N$. Define $\mathcal{R} \subseteq E \times E$ as*

$$\mathcal{R} = \{(x, y) : |x - y| \leq a\}.$$

$\mathcal{R}$ is reflexive, symmetric, but not transitive.

(iii) *Let $E = Z$, $m \in N$. Define the congruence $\bmod m$ on $E \times E$ as follows:*

$$x \equiv y \text{ if } (\exists k \in \mathbb{Z} : x - y = km).$$

This is an equivalence relation on E.

Definition 1.2.7 *Let $\mathcal{R}$ be an equivalence relation on E.*

1. *The equivalence class of an element x in E is the set of all elements $y \in E$ that are in relation with x we denote this set by . $\dot{x}$ or $\bar{x}$ or $\mathcal{C}(x)$,and we write it as follow*

$$\dot{x} = \bar{x} = \mathcal{C}(x) = \{y \in E : y\mathcal{R}x\}.$$

2. *$\bar{x}$ is a representative of the equivalence class $\mathcal{C}(x)$.*

3. *The set of equivalence classes for all elements in E is called the "quotient set" of E for the equivalence relation $\mathcal{R}$. It is denoted as $E/\mathcal{R}$, and written as follows:*

$$E/\mathcal{R} = \{\mathcal{C}(x) : x \in E\}.$$

Example 1.2.8 *In $\mathbb{R}$ we define the relation $\mathcal{R}$ by:*

$$x\mathcal{R}y \Leftrightarrow x - y \in \mathbb{Z}.$$

This relation is indeed a relation of equivalence. Indeed,

- *For $x \in \mathbb{R} : x\mathcal{R}x \Leftrightarrow 0 \in \mathbb{Z}$, as $0 \in \mathbb{Z}$, then $x\mathcal{R}x, \forall x \in \mathcal{R}$, so $\mathcal{R}$ is a reflexive relation.*
- *For $x, y \in \mathbb{R}$, we have $(x\mathcal{R}y) \Leftrightarrow (x - y \in \mathbb{Z}) \Leftrightarrow (y - x \in \mathbb{Z}) \Rightarrow y\mathcal{R}x$, then $\mathcal{R}$ is a symmetric relation.*
- *For $x, y, z \in \mathbb{R}$, we have*

$$\begin{aligned}(x\mathcal{R}y \text{ and } y\mathcal{R}z) &\Rightarrow (x-y\in\mathbb{Z} \text{ and } y-z\in\mathbb{Z})\\ &\Rightarrow (x-y+y-z\in\mathbb{Z})\\ &\Rightarrow (x-z\in\mathbb{Z})\Rightarrow(x\mathcal{R}z),\end{aligned}$$

then $\mathcal{R}$ is a transitive relation.

Therefore, the set of equivalence classes $\mathcal{C}(x)$ is the set

$$\begin{aligned}\mathcal{C}(x) &= \{y\in\mathbb{R} : y-x\in\mathbb{Z}\}\\ &= \{y\in\mathbb{R} : y\in x+\mathbb{Z}\}\\ &= \{y\in\mathbb{R} : y=k+x : k\in\mathbb{Z}\}\\ &= \{k+x : k\in\mathbb{Z}\},\end{aligned}$$

if $x\in\mathbb{Z}$, we have $\mathcal{C}(x)=\mathbb{Z}$.

Example 1.2.9 *Let us consider the relation $\mathcal{R}$ defined on $\mathbb{R}$ by :*

$$\forall x,y\in\mathbb{R},\ x\mathcal{R}y \Leftrightarrow xe^y = ye^x$$

Prove that $\mathcal{R}$ is an equivalence relation.

Solution 1.2.10 *show that $\mathcal{R}$ is reflexive, symmetric and transitive.*

1.$\forall x\in\mathbb{R}$ *on a* $xe^x = xe^x$. *In other words, we have $x\mathcal{R}x$ and then $\mathcal{R}$ is reflexive.*

2. *$\mathcal{R}$ is symetric . In fact, let x, $y\in\mathbb{R}$, such that $x\mathcal{R}y$, hence we have*

$$\begin{aligned}x\mathcal{R}y &\Rightarrow xe^y = ye^x,\\ &\Rightarrow ye^x = xe^y,\\ &\Rightarrow y\mathcal{R}x,\end{aligned}$$

3. *$\mathcal{R}$ is transitive because for all x, y, $z\in\mathbb{R}$, such that $[(x\mathcal{R}y)\wedge(y\mathcal{R}z)]$, on a :*

$$x\mathcal{R}y \Rightarrow xe^y = ye^x................(1)$$
$$y\mathcal{R}z \Rightarrow ye^z = ze^y................(2)$$

(2) gives $y=\dfrac{ze^y}{e^z}$, moreover, using (1) and by substituting y we have $xe^y = \dfrac{ze^y}{e^z}e^x$ hence $xe^ye^z = ze^ye^x$. Since $e^y\neq 0$ Thus $xe^z = ze^x$, which implies $x\mathcal{R}z$.

4. *$\mathcal{R}$ is reflexive, symetric and transitive then it is an equivalence relation.*

1.2.3 Order Relation

Definition 1.2.11 *A binary relation $\mathcal{R}$ over E is said to be an order relation if it is antisymmetric, transitive and reflexive*

Example 1.2.12 *On $\mathbb{R}$ the relation $\leq$ is an order relation. In fact*
- $\mathcal{R}$ *reflexive*

$$\forall x \in \mathbb{R}, x\mathcal{R}z \Leftrightarrow x = x.$$

- $\mathcal{R}$ *antisymmetric, if only if:*

$$\forall (x, y) \in \mathbb{R} \times \mathbb{R}; (x\mathcal{R}y \Leftrightarrow x \leq y) \text{ and } (y\mathcal{R}x \Leftrightarrow y \leq x) \Leftrightarrow x = y.$$

-$\mathcal{R}$ *transitive, if only if :*

$$\forall (x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R};\ (x\mathcal{R}y \Leftrightarrow x \leq y) \text{ and } (y\mathcal{R}z \Leftrightarrow y \leq z) \Leftrightarrow x \leq z \Leftrightarrow x\mathcal{R}z.$$

- *In $\mathbb{R}$, the relation $<$ is not a relation of order (is not reflexive.)*

1.2.4 Total order and partial order

Definition 1.2.13 *Let $\mathcal{R}$ be a relation of order defined on a set E, we say that $\mathcal{R}$ is total, if for all $x, y \in E$, we have*

$$x\mathcal{R}y \text{ or } y\mathcal{R}x.$$

Otherwise, we say that $\mathcal{R}$ is partial, i.e.

$$\exists x, y \in E : \text{neither } x\mathcal{R}y \text{ nor } y\mathcal{R}x$$

Example 1.2.14 $A = \{1, 2, 3, 4, 5, 6\}$ *with*

$$a\mathcal{R}b \Leftrightarrow a \text{ divide } b$$

is a partial order relation (is not total)

Indeed 2 *and* 3*, for example, are not comparable* : 2 *does not divide* 3 *and* 3 *does not divide* 2.

Example 1.2.15 *Let A be a non-empty set and $\mathcal{R}$ a relation on A defined by :*

$$\forall a, b \in A, a\mathcal{R}b \Leftrightarrow a = b.$$

$\mathcal{R}$ is a an order relation on A.

If A is a singleton, then the order is total. If not, the order is partial.

1.3 Applications

1.3.1 Denition of an application

Definition 1.3.1 *Let E and F be given sets, we call the application of E in F, any correspondence f between the elements of E and those of F which associates to any element of E one and only element of F, we write*

$$\begin{aligned} f: \quad & E \to F \\ & x \longrightarrow f(x) \end{aligned}$$

or f (application)$\Leftrightarrow (\forall x \in E)(\exists! y \in F) : y = f(x)$

The set E is said to be the starting set and F is said to be the end set.
The element x is said to be the antecedent and y is said to be the image of x by f.
The map f is said to be a function if, for each $x \in E$, there exists at most $y \in F$ such that $f(x) = y$.

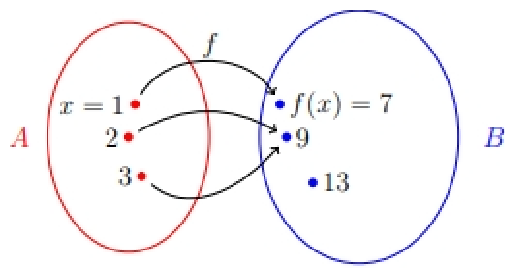
Remark 1.3.2 ***(1)*** *The application from E to F is that $\Leftrightarrow$ every element x of E has a unique image in F.*

(2) *If f is an application from E to F, then the element y of F can have more than one precedent in E.*

(3) *We must defferentiate between $f(x)$ and f : we have $f(x) \in F$, while f represents the application as a whole, and it belongs to the space of applications defened from E to F.*

Example 1.3.3 *We have $A = \{1, 2, 3\}$ and $B = \{7, 9, 13\}$.*

- *We have $f(3) = 9$, $f(2) = 9$; $f(1) = 7$.*
- *f application from A to B every element x of A has a unique image in B.*



- *This element 13 has no precedent according to the application.*
- *This element 9 has two precedent : 2 and 3.*

Definition 1.3.4 *(Graph). Let E and F be given sets. The graph of a map $f : E \to F$ is*

$$G_f := \{(x, f(x)) : x \in E\} \subset E \times F.$$

Definition 1.3.5 *(Equality). Let $f, g : E \to F$ be the applications. We say that f, g are equal if and only if for all $x \in E : f(x) = g(x)$. We then write $f = g$.*

Definition 1.3.6 *(Composition). Let E, F and G be three sets and f and g two maps such as*

$$E \xrightarrow{f} F \xrightarrow{g} G$$

We can deduce from this a map of E in G denoted $g \circ f$ and called a map composed of f and g, by

$$(g \circ f)(x) = g(f(x)), \quad \text{for all} \quad x \in E.$$

Example 1.3.7 *Let $f : Z \to Z$, $g : Z \to Z$,*

$$f(x) = x^2 + 2, \; g(x) = 2x - 1.$$

Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution : we have

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) = g(x)^2 + 2 = 4x^2 - 4x + 3, \\ (g \circ f)(x) &= g(f(x)) = 2f(x) - 1 = 2x^2 + 3. \end{aligned}$$

As you clearly see from the above,$f \circ g \neq g \circ f$ in general.

Definition 1.3.8 *Let E be a set, we call an identity map, denoted $Id_E : E \to E$ is the map that verifies $Id_E(x) = x$, $\forall x \in E$.*

Definition 1.3.9 *Let $f : E \to F$ be a function. The domain of denition of f, denoted D_f, is the set of elements $x \in E$ in which there exists a single element $y \in F$, such that $y = f(x)$.*

Example 1.3.10 *Let $f : \mathbb{R} \to \mathbb{R}$ denied by $f(x) = \sqrt{x+1}$, then*

$$D_f = \{x \in \mathbb{R} : x + 1 \geq 0\} = [-1, +\infty[.$$

1.3.2 Restricting and extending an application

Definition 1.3.11 *Let $A \subset E$ and $f : E \to F$ be an application. We call the restriction from f to A, the map $f_{/A} : E \to F$ denied by*

$$f_{/A}(x) = f(x), \text{ for all } x \in A.$$

Definition 1.3.12 *Let $E \subset G$ and $f : E \to F$ a map. We call an extension from f to G, any map g from G to F whose restriction to E is f.*

Example 1.3.13 *Given the applicationf:*

$$\begin{array}{ll} f : \mathbb{R}_+^* & \to R \\ \quad x & \to \ln x \end{array},$$

then

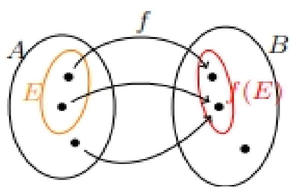
$$\begin{array}{ll} g : \mathbb{R}^* & \to R \\ \quad x & \to \ln |x| \end{array}, \qquad \begin{array}{ll} h : \mathbb{R}^* & \to R \\ \quad x & \to \ln (|3x| - 2x) \end{array},$$

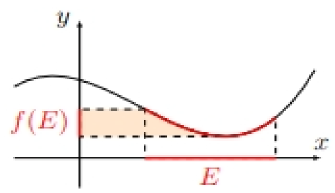
are two different extensions of f to $\mathbb{R}^*$.

1.3.3 Direct image and inverse image

Definition 1.3.14 *Let A , B be non-empty sets. Let E be a subset of A, and $f : A \to B$ be application. The direct image of the set E is defined by :*

$$f(E) = \{f(x) : x \in E\}$$





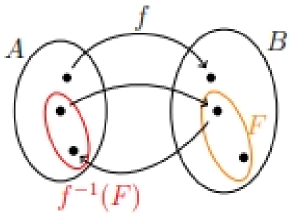
Example 1.3.15 *Let $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$. Let*

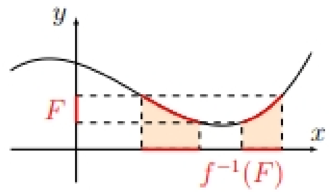
$$A = \{x \in \mathbb{Z} : 0 \leq x \leq 2\}.$$

Then $f(A) = \{0, 1, 4\}$.

Definition 1.3.16 *Let A , B be non-empty sets, let F be a subset of B, and $f : A \to B$ be application. The inverse image of the set F is defined by :*

$$f^{-1}(F) = \{x \in A : \ f(x) \in F\}$$





Example 1.3.17 *Let* $f : Z \to Z$ *defined by* $f(x) = x^2$, *let* $B = \{y \in Z : y \leq 10\}$. *Then* $f^{-1}(B) = \{-3, -2, -1, 0, 1, 2, 3\}$.

theorem 1.3.18 *Let* $f : X \to Y$ *and* $A_1 \subset X$, $A_2 \subset X$, $B_1 \subset Y$, $B_2 \subset Y$. *Then*

(i)) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$ *and* $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(ii) $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ *and* $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(iii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ *and* $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.

(iiii) $A_1 \subset f^{-1}(f(A_1))$ *and* $f\left(f^{-1}(B_1)\right) \subset B_1$.

1.3.4 Injective, surjective and bijective application

Definition 1.3.19 *Let* $f : E \to F$. f *is said to be injective if and only if :*

$$\forall (x_1, x_2) \in E^2 : f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Example 1.3.20

$$\begin{aligned} f : \mathbb{R}^+/\{2\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \frac{1}{x^2 - 4} \end{aligned}$$

is an injective application because we have :

$$\forall (x_1, x_2) \in \left(\mathbb{R}^+/\{2\}\right)^2 : \; f(x_1) = f(x_2) \Leftrightarrow \frac{1}{x_1^2 - 4} = \frac{1}{x_2^2 - 4} \Leftrightarrow x_1^2 = x_2^2 \Leftrightarrow x_1 = \pm x_2,$$

but as x_1, $x_2 \in \mathbb{R}^+/\{2\}$ *then* $x_1 = x_2$.

Definition 1.3.21 *Let* $f : E \to F$. *We say that* f *is surjective if and only if: for all* $y \in F$, *there exists* $x \in E$ *such that* $f(x) = y$, *i.e.*

$$\forall y \in F, \exists x \in E : \; y = f(x).$$

Example 1.3.22 *Let* $f : \mathbb{Z} \to \mathbb{N}$.*be the map defined by* $f(x) = |x|$, *then* f *is surjective. Indeed, let* $y \in \mathbb{N}$, *for* $x = y$ *or* $x = -y$, *we have* $x \in \mathbb{Z}$ *and* $f(x) = |x| = y$, *so there exists* $x \in \mathbb{Z}$ *such that* $y = f(x)$.

Definition 1.3.23 *Let $f : E \to F$. f is said to be bijective if and only if: f is both injective and surjective. This is equivalent to : for all $y \in F$ there exists a unique $x \in E$ such that $y = f(x)$. In other words:*

$$\forall y \in F, \exists! x \in E : y = f(x).$$

Example 1.3.24 *Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x + 1$, then f is bijective. Indeed, let $y \in \mathbb{R}$, such that $f(x) = y$, then $x = y - 1$, so there exists a unique x in $\mathbb{R}$ such that $y = f(x)$.*

Remark 1.3.25 *If the application f is bijective, then to every $y \in F$ we match a single element $x \in E$.*

Definition 1.3.26 *Let $f : E \to F$ be a bijective function. We define the function $f^{-1} : F \to E$, called the reciprocal function of f, given by $f^{-1}(x) = y$ if and only if $f(y) = x$.*

Example 1.3.27 *Let f be the map defined by $f(x) = x^2 + 1$ of $\mathbb{R}^+ \to [1, +\infty[$, then f is bijective, because for all $y \in [1, \infty[$, the equation $y = f(x)$ admits a single solution $x = \sqrt{y-1}$. The reciprocal bijection is $f^{-1} : [1, +\infty[\to \mathbb{R}+$ defined by:*

$$f^{-1}(x) = \sqrt{x-1} \text{ for all } x \in [1, +\infty[.$$

Proposition 1.3.28 *Let E, F be sets and $f : E \to F$ an application.*

- *The map f is bijective if and only if there is a map $g : F \to E$ such that*

$$f \circ g = Id_F \text{ and } g \circ f = Id_E.$$

- *Let $f : E \to F$ and $g : F \to G$ be bijective maps. The map $g \circ f$ is bijective and its reciprocal bijection is*

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

1.4 Some methods of proof

1. First we discuss a couple of widely used methods of proof: contrapositive proof and proof

by contradiction.

The idea of contrapositive proof is the following equivalence

$$(A \Rightarrow B) \Leftrightarrow (\bar{B} \Rightarrow \bar{A}).$$

So to prove that$A \Rightarrow B$ is true is the same as to prove that $\bar{B} \Rightarrow \bar{A}$ is true.

Example 1.4.1 *For integers m and n, if mn is odd then so are m and n.*

proof. We have to prove that

$$(\forall m, n \in \mathbb{Z}_{+})(mn \text{ is odd}) \Rightarrow [(m \text{ is odd}) \wedge (n \text{ is odd})],$$

which is the same as to prove that

$$[(m \text{ is even}) \vee (n \text{ is even})] \Rightarrow (mn \text{ is even})$$

The latter is evident. ■

The idea of proof by contradiction is the following equivalence

$$(A \Rightarrow B) \Leftrightarrow (\bar{A} \vee B) \Leftrightarrow \overline{(A \wedge \bar{B})}$$

So to prove that $A \Rightarrow B$ is true is the same as to prove that $\bar{A} \vee B$ is true or elsethat $A \wedge \bar{B}$ is false.

2 The Principle of Mathematical Induction is often used when one needs to prove statements of the form

$$(\forall n \in \mathbb{N})\ P(n).$$

Thus one can show that 1 has property P and that whenever one adds 1 to a number that has property P, the resulting number also has property P.

Principle of Mathematical Induction. If for a statement $P(n)$

(i) $P(1)$is true,

(ii) $[P(n) \Rightarrow P(n+1)]$ is true,

then $(\forall n \in N)\ P(n)$ is true.

Part **(i)** is called the base case; **(ii)** is called the induction step.

Example 1.4.2 *Prove that*

$$\forall n \in N : 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution: Base case: $n = 1$. $1^2 = \frac{1{\cdot}2{\cdot}3}{6}$ *is true.*

Induction step: Suppose that the statement is true for $n = k(k \geq 1)$. *We have to prove thatit is true for* $n = k + 1$. *So our assumption is*

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}.$$

Therefore we have1

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6},$$

which proves the sta tement for $n = k + 1$. *By the principle of mathematical induction thestatement is true for all* $n \in \mathbb{N}$.