Ferhat Abbas University-Setif 1
Faculty of Sciences
Department of Chemistry/Physics

Solved Exercises

Exercise 1.

a- Determine by the trapezoidal and Simpson methods an approximate value of the integral $\int_0^2 f(x)dx$ using the data of the following table :

x_i	0	1/2	1	1.5	2
$f(x_i) = e^{x^2}$	1	1.284	2.718	9.487	54.598

b- Estimate the calculation error in each case using 10 subdivision.

c- What is the number of required subdivision to achieve a precision of $\epsilon(I) = 10^{-2}$ for each method ?

Solution .

a.1- The approximation of the integral by the trapezoidal method.

$$\begin{aligned}\int_0^2 f(x)dx \simeq I(f) &= \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1}(f(x_i))\right]\\ &= \frac{0.5}{2}[1 + 54.598 + 2(1.284 + 2.718 + 9.487)]\\ &= 20.644\end{aligned}$$

a.2- Approximating the integral using the Simpson.

$$\begin{aligned}\int_0^2 f(x)dx \simeq I(f) &= \frac{h}{3}\left[f(0) + f(2) + 2\sum_{i=1}^{k-1} f(x_{2i}) + 4\sum_{i=1}^{k} f(x_{2i-1})\right]\\ &= \frac{0.5}{3}[f(0) + f(2) + 2f(1) + 4(f(0.5) + f(1.5))]\\ &= \frac{0.5}{3}(1 + 54.598 + 2 \times 2.718 + 4 \times (1.284 + 9.487))\\ &= 26.0295\end{aligned}$$

b.1- Evaluating the error by the trapezoidal method with $n = 10$.
We have $f''(x) = 2e^{x^2} + 4x^2e^{x^2}$ and $\max_{x\in[0,2]} |f''(x)| = 982.766$, thus :

$$\begin{aligned}\mathcal{R}_n(f) &\le \frac{(b-a)^3}{12n^2}\max_{x\in[a,b]} |f''(x)|\\ &\le \frac{8}{12 \times 10^2}982.766 = 6.551\end{aligned}$$

b.2- Evaluating the error by Simpson method with $n = 10$.
We have $f^{(4)}(x) = 12e^{x^2} + 48x^2e^{x^2} + 16x^4e^{x^2} = e^{x^2}(16x^4 + 48x^2 + 12)$ and $\max_{x\in[0,2]} |f^{(4)}(x)| = |f^{(4)}(2)| = 39092.275$, thus :

$$\begin{aligned}\mathcal{R}_n(f) &\le \frac{(b-a)^5}{180n^4}\max_{x\in[a,b]} |f^{(4)}(x)|\\ &\le \frac{2}{180 \times 10^4}39092.275 = 0.6950\end{aligned}$$

c.2- Calculating the number of points required to achieve a precision 10^{-2} with the trapezoidal method. We have :

$$n \geq \sqrt{\frac{(b-a)^3 \max\limits_{x\in[a,b]} \left|f''(x)\right|}{12\,\varepsilon}}$$
$$\geq \sqrt{\frac{982.766 \times 2^3}{12 \times 10^2}} \simeq 256$$

Thus, the number of required subdivisions is $n \geq 256$.

c.1- Calculating the number of required subdivision to achieve a precision of 10^{-2} with Simpson method. We have :

$$n \geq \sqrt[4]{\frac{(b-a)^5 \max\limits_{x\in[a,b]} \left|f^{(4)}(x)\right|}{180\varepsilon}}$$
$$\geq \sqrt[4]{\frac{39092.275 \times 2^5}{180 \times 10^{-2}}} \simeq 51.344$$

Thus, the number of required subdivisions using Simpson method is $n \geq 52$.

Exercise 2. Consider the following integral $I = \displaystyle\int_0^{\pi} \sin(x)dx$.

1. Calculate the exact value of I.

2. Using the trapezoidal and Simpson methods with $h = \frac{\pi}{4}$:

a- Approximate the value of the integral I.

b- Estimate the calculation error.

c- Evaluate the absolute error.

3. Find the value of steplenght h and the number of required subdivisions so that the error obtained by the trapezoidal (resp. Simpson) method is less than 5×10^{-4}.

Solution .

1. $I = \displaystyle\int_0^{\pi} \sin(x)dx = 2$.

2. I- Approximating the integral using the trapezoidal method :

a-

$$I(f) = \frac{h}{2}\left[f(x_0) + f(x_n) + 2\sum_{i=1}^{3} f(x_i)\right]$$
$$= \frac{\pi}{8}(f(0) + f(\pi) + 2(f(\pi/4) + f(\pi/2) + f(3\pi/4)))$$
$$\simeq 1.896.$$

b- We have

$$\mathcal{R}_n(f) \leq \frac{(b-a)^3}{12n^2} \max_{x\in[a,b]} \left|f''(x)\right| = \frac{b-a}{12} h^2 \max_{x\in[a,b]} \left|f''(x)\right|$$
$$\leq \frac{\pi^3}{192} \max_{x\in[0,\pi]} \left|\sin(x)\right| \leq \frac{\pi^3}{192} \simeq 0.16149.$$

c- $\left|I(f) - \displaystyle\int_0^{\pi} \sin(x)dx\right| = 0.1038$

2. II- The Simpson method :

a-

$$I(f) = \frac{h}{3}\left[f(a) + f(b) + 2\sum_{i=1}^{k-1} f(x_{2i}) + 4\sum_{i=1}^{k} f(x_{2i-1})\right], \text{ with } k = 2$$
$$= \frac{\pi}{12}(f(0) + f(\pi) + 2f(\frac{\pi}{2}) + 4(f(\frac{\pi}{4}) + f(\frac{3\pi}{4}))) \simeq 2.04.$$

b- We have

$$\begin{aligned}\mathcal{R}_n(f) &\leq \frac{(b-a)^5}{180n^4}\max_{x\in[a,b]}\left|f^{(4)}(x)\right| \\ &\leq \frac{\pi^5}{180\times 4^4}\max_{x\in[0,\pi]}\left|\sin(x)\right| \leq \frac{\pi^5}{180\times 4^4} \simeq 0.0066\end{aligned}$$

c- $|I(f) - \int_0^{\pi} \sin(x)dx| = 0.004$

3. Let's calculate the value of the steplenght h and the number of required subdivisions n so that the error is less than $\varepsilon = 5\times 10^{-4}$.

3.I- The trapezoidal method :

We have

$$\begin{aligned}n &\geq \sqrt{\frac{(b-a)^3 \max_{x\in[a,b]}\left|f''(x)\right|}{12\,\varepsilon}} \\ &\geq \sqrt{\frac{\pi^3 \max_{x\in[0,\pi]}\left|\sin(x)\right|}{12\times 5\times 10^{-4}}} \simeq 71.8\end{aligned}$$

Thus, the number of required subdivisions is $n \geq 72$.

3.II- The Simpson method :

We have

$$\begin{aligned}n &\geq \sqrt[4]{\frac{(b-a)^5 \max_{x\in[a,b]}\left|f^{(4)}(x)\right|}{180\varepsilon}} \\ &\geq \sqrt[4]{\frac{\pi^5 \max_{x\in[0,\pi]}\left|\sin(x)\right|}{180\times 5\times 10^{-4}}} \simeq 7.64116\end{aligned}$$

Thus, the number of required subdivisions using Simpson method is $n \geq 8$.

Exercise 3. A rocket is launched vertically from the ground, and its acceleration γ is measured during the first 80 seconds :

t in (s)	0	10	20	30	40	50	60	70	80
γ in m/s^2	30	31.63	33.44	35.47	37.75	40.33	43.29	46.70	50.67

- Calculate the velocity V of the rocket at $t = 80s$, using the trapezoidal and Simpson methods.

Solution .

We know that the acceleration γ is the derivative of the velocity V, so

$$V(t) = V(0) + \int_0^t \gamma(t)dt = 0 + \int_0^{80} \gamma(t)dt$$

- First, lets calculate $V(80)$ using the trapezoidal method. According to the previous table of values, we have $h = 10$ and $n = 8$. Therefore,

$$\begin{aligned}V(80) &= \frac{h}{2}\left(\gamma(t_0) + \gamma(t_2) + 2(\gamma(t_1)\cdots + \gamma(t_7))\right) \\ &= 5(30 + 50.67 + 2(31.63 + 33.44 + 35.47 + 37.75 + 40.33 + 43.29 + 46.70)) \\ &= 3089\,\mathrm{m/s}.\end{aligned}$$

- Next, we calculate $V(80)$ using Simpson method :

$$\begin{aligned}V(80) &= \frac{h}{3}\left[\gamma(t_0) + \gamma(t_k) + 2\sum_{i=1}^{3}\gamma(t_{2i}) + 4\sum_{i=1}^{4}\gamma(t_{2i-1})\right] \\ &= \frac{10}{3}(30 + 50.67 + 2(33.44 + 37.75 + 43.29) + 4(31.63 + 35.47 + 40.33 + 46.70)) \\ &= 3087\,\mathrm{m/s}\end{aligned}$$

Exercise 4.

a- Using 4 subintervals, determine by the trapezoidal method an approximate value of $\int_0^2 \sin^2(x)dx$, and estimate the calculation error.

b- What is the number of required subintervals to achieve a precision of 10^{-2} by the trapezoidal method ?

Solution .

a- Approximating the integral using the trapezoidal method :

$$\begin{aligned}\int_0^2 f(x)dx \simeq I(f) &= \frac{h}{2}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1}(f(x_i))\right]\\ &= \frac{0.5}{2}[f(0)+f(2)+2(f(0.5)+f(1)+f(1.5))]\\ &= 1.173\end{aligned}$$

- Evaluating the error by the trapezoidal method with $n=4$: We have $f''(x) = 2(\cos^2(x)-\sin^2(x)) = 2(1-2\sin^2(x))$ and $\max_{x\in[0,2]}\left|f''(x)\right| = f''(0) = f''(\frac{\pi}{2}) = 2$, thus :

$$\begin{aligned}\mathcal{R}_n(f) &\le \frac{(b-a)^3}{12n^2}\max_{x\in[a,b]}\left|f''(x)\right|\\ &\le \frac{8}{12\times 4^2}2 = 0.0833\end{aligned}$$

b- The number of required subdivisions to achieve a precision of 10^{-3} with the trapezoidal method.
We have :

$$\begin{aligned}n &\ge \sqrt{\frac{(b-a)^3\max_{x\in[a,b]}\left|f''(x)\right|}{12\,\varepsilon}}\\ &\ge \sqrt{\frac{2\times 2^3}{12\times 10^{-3}}} \simeq 36.51\end{aligned}$$

Thus, the number of required subdivisions is $n \ge 37$.

Exercise 5. We consider the following integral :

$$I = \int_0^1 \frac{dx}{1+x^2}dx.$$

1. Calculate the exact value of this integral.
2. Approximate the value of this integral numerically using :
- the midpoint rule with 5 intervals.
- the trapezoidal method with 4 intervals.
- Simpson method with 2 intervals.

Abbreviated Solution .

1. $I = \int_0^1 \frac{dx}{1+x^2} = 0.7854$.
2.
- By the midpoint rule, we obtain : $I(f) = 0.8387$
- By the trapezoidal method, we obtain : $I(f) = 0.7828$
- By Simpson method, we obtain : $I(f) = 0.7854$

Exercise 6. Determine the required number of subdivisions to approximate the integral $\int_0^1 xe^{-x}dx$ with a precision of 10^{-8} using :
1. Trapezoidal method.
2. Simpson method.

Abbreviated Solution .

1. $n \geq 4083$
2. $n \geq 40$

Exercise 7. Determine the number of required subdivisions to approximate the value of the integral $\int_{-\pi}^{\pi} \cos(x)dx$ with a precision of 5×10^{-4} using Simpson method.

Abbreviated Solution .

- $n \geq 20$