وزارة التعليـم العالـي والبحـث العلمـي

**Sétif 1 University-Ferhat ABBAS**
**Faculty of Sciences**
**Department of Mathematics**

# Introduction to Metric and Topological Spaces

*Mathematics Bachelor's Degree - LMD - $3^{rd}$ Semester*

Series 1 (Usual topology on $\mathbb{R}$)

**Exercise 1:** We consider in $\mathbb{R}$ the following family:

$$\mathcal{T} = \{A \subseteq \mathbb{R} \mid (A = \emptyset) \text{ or } (\forall x \in A, \exists I_x \text{ such that } x \in I_x \subseteq A)\},$$

where $I_x$ is an open interval in $\mathbb{R}$.

**1)** Let $A \in \mathcal{T}$ and $A \neq \emptyset$, show that $A = \bigcup_{x \in A} I_x$.

**2)** Show that $\mathcal{T}$ is a topology on $\mathbb{R}$ (the usual topology on $\mathbb{R}$).

**3)** Show that $]a, b[\,, ]a, +\infty[$ and $]-\infty, b[$ are open sets.

**4)** Show that $[a, b]\,, [a, +\infty[$ and $]-\infty, b]$ are closed sets.

**5)** Show that $]a, b]$ and $[a, b[$ are neither closed nor open sets.

**6)** Show that $(\mathbb{R}, |.|)$ is a Hausdorff space (separated).



**Exercise 2:** Show that any finite union of closed sets in $\mathbb{R}$ is an closed set in $\mathbb{R}$.

**Exercise 3:** Show that any intersection of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$.

**Exercise 4:** Find the accumulation points and the isolated points of each of the following sets of real numbers:

**1)** $A_1 = \mathbb{N}$, **2)** $A_2 = ]a, b]$,
**3)** $A_3 = \complement_{\mathbb{R}} Q$, **4)** $A_4 = ]-\infty, -1[\cup]-1, 3[\cup\{5\}$,
**5)** $A_5 = \left\{(-1)^n + \frac{1}{2^n} : n \in \mathbb{N}\right\}$.

**Exercise 5:** Find sets $A$ such that:

**1)** $A$ and $A'$ are disjoint, **2)** $A$ is a proper subset of $A'$,
**3)** $A'$ is a proper subset of $A$, **4)** $A = A'$.

**Exercise 6:** Let $(x_n)$ be a Cauchy sequence. If a subsequence $(x_{n_k})$ of $(x_n)$ converges to a point $\ell$, then the Cauchy sequence itself converges to $\ell$.

**Exercise 7:** Let $A$ be a subset of $\mathbb{R}$. A point $x_0 \in \mathbb{R}$ is an accumulation point of $A$ if and only if every neighborhood of $x_0$ contains an infinite number of points from $A$.

**Exercise 8:** A subset $A$ of $(\mathbb{R}, |.|)$ is closed if and only if it contains all its accumulation points.