وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solution of series 1 (Usual topology on $\mathbb{R}$)

Exercise 1: We consider in $\mathbb{R}$ the following family:

$$\mathcal{T} = \{A \subseteq \mathbb{R} \mid (A = \emptyset) \text{ or } (\forall x \in A,\ \exists I_x \text{ such that } x \in I_x \subseteq A)\},$$

where I_x is an open interval in $\mathbb{R}$.

1) Let $A \in \mathcal{T}$ and $A \neq \emptyset$. On the one hand, for all $x \in A$, there exists $I_x \subseteq A$ such that $x \in I_x \subseteq A$, from which we obtain

$$\bigcup_{x \in A} I_x \subseteq A. \tag{1}$$

On the other hand, we have

$$A = \bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} I_x. \tag{2}$$

From (1) and (2), we conclude that $A = \bigcup_{x \in A} I_x$.

2) From the previous question we deduce that for each $x \in A$ there is $r > 0$ such that $x \in B(x,r) =]x-r, x+r[\subseteq I_x \subseteq A$. Then, it follows from Definition(1.18) that $\mathcal{T}$ is a topology on $\mathbb{R}$.

3)
- The interval $]a,b[$ is open in $(\mathbb{R}, |.|)$ because for each $x \in]a,b[$ there exists $r > 0$ such that $B(x,r) =]x-r, x+r[\subseteq]a,b[$ (By Definition(1.4)).
- The interval $]a,+\infty[$ is open in $(\mathbb{R}, |.|)$ because for each $x \in]a,+\infty[$ there exists $r > 0$ such that $B(x,r) =]x-r, x+r[\subseteq]a,+\infty[$ (By Definition(1.4)).
- The interval $]-\infty,b[$ is open in $(\mathbb{R}, |.|)$ because for each $x \in]-\infty,b[[$ there exists $r > 0$ such that $B(x,r) =]x-r, x+r[\subseteq]-\infty,b[$ (By Definition(1.4)).

4) • The interval $[a,b]$ is closed in $(\mathbb{R},|.|)$ because $C_{\mathbb{R}}[a,b] =]-\infty,a[\bigcup]b,+\infty[$ is an open set in $(\mathbb{R},|.|)$.

• The interval $[a,+\infty[$ is closed in $(\mathbb{R},|.|)$ because $C_{\mathbb{R}}[a,+\infty] =]-\infty,a[$ is an open set in $(\mathbb{R},|.|)$.

• The interval $]-\infty,b]$ is closed in $(\mathbb{R},|.|)$ because $C_{\mathbb{R}}]-\infty,b] =]b,+\infty[$ is an open set in $(\mathbb{R},|.|)$.

5) • The interval $]a,b]$ is not open because there does not exist an $I_b \subset \mathbb{R}$ such that $b \in I_b \subseteq]a,b]$, and it is not closed because the complement $C_{\mathbb{R}}]a,b] =]-\infty,a]\cup]b,+\infty[$ is not open.

• Using similar arguments to those in (4), we can show that $[a,b[$ is neither open nor closed.

6) We want to show that $(\mathbb{R},|.|)$ is a Hausdorff space (separated).
Let $x,y \in \mathbb{R}$ such that $x \neq y$. We set $|x-y| = r$. If we take $N_1 = \left]x-\frac{r}{3}, x+\frac{r}{3}\right[$ as a neighbourhood of x and $N_2 = \left]y-\frac{r}{3}, y+\frac{r}{3}\right[$ as a neighbourhood of y, we obtain $N_1 \cap N_2 = \emptyset$. Hence, $(\mathbb{R},|.|)$ is a Hausdorff space (separated).

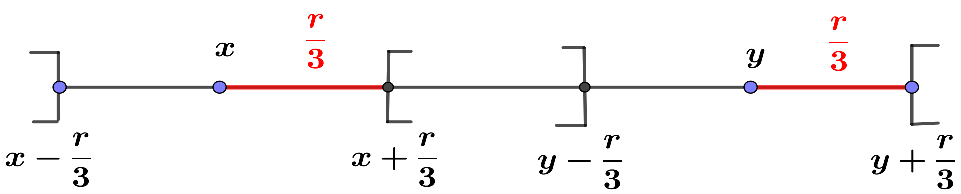

Figure 1: $\mathbb{R}$ is a Hausdorff space

Exercise 2: Let $F_1, F_2, \ldots, F_n$ be closed sets in $\mathbb{R}$ and let $F = F_1 \cup F_2 \cup \cdots \cup F_n$. Then, according to De Morgan's law, we obtain $F^C = (F_1 \cup F_2 \cup \cdots \cup F_n)^C = F_1^C \cap F_2^C \cap \cdots \cap F_n^C$. Thus, F^C is open because it is the finite intersection of open sets, which implies that F is a closed set in $\mathbb{R}$.

Exercise 3: Let $\{F_i, i \in I\}$ be a family of closed sets in $\mathbb{R}$ and let $F = \bigcap_{i\in I} F_i$. Then, according to De Morgan's law, we obtain $F^C = \left(\bigcap_{i\in I} F_i\right)^C = \bigcup_{i\in I} F_i^C$. Thus, F^C is open because it is an any union of open sets, which implies that F is a closed set in $\mathbb{R}$.

Exercise 4:

1) $A_1' = \emptyset,\quad Is(A_1) = \mathbb{N}$ **2)** $A_2' = [a,b]\,,\quad Is(A_2) = \emptyset.$

3) $A_3' = \mathbb{R},\quad Is(A_3) = \emptyset$ **4)** $A_4' =]-\infty, 3],\quad Is(A_5) = \{5\}$

5) $A_5' = \{-1, 1\}\,,\quad Is(A_5) = A_5.$

Exercise 5:

1. The set $A = \left\{\frac{1}{n} : n \geqslant 1\right\}$ has 0 as its only accumulation point. Hence $A \bigcap A' = \emptyset$.
2. Let $A =]a,b]$ with $a, b \in \mathbb{R}$. As we saw in the previous exercise, $A' = [a,b]$. Thus, $A \subset A'$.
3. Let $A = \left\{\frac{1}{n} : n \geqslant 1\right\} \cup \{0\}$. Then, $A' = \{0\}$. Therefore, $A' \subset A$.
4. Let $A = [a,b] \subset \mathbb{R}$. Then, $A' = [a,b]$ which implies that $A' = A$.

Exercise 6: Let $\varepsilon > 0$. We need to find a positive integer N such that

$$n > N \implies |x_n - \ell| < \varepsilon.$$

Since (x_n) is a Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n, m \in \mathbb{N},\quad n, m > n_0 \implies |x_n - x_m| < \frac{\varepsilon}{2}.$$

Moreover, (x_{n_k}) is a subsequence that converges to ℓ, hence there exists $n_1 \in \mathbb{N}$ such that

$$n_k > n_1 \implies |x_{n_k} - \ell| < \frac{\varepsilon}{2}.$$

If we take $N = \max(n_0, n_1)$ and $n_k, n > N$, we obtain

$$|x_n - \ell| \leqslant |x_n - x_{n_k} + x_{n_k} - \ell| \leqslant |x_n - x_{n_k}| + |x_{n_k} - \ell| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which means that $\lim\limits_{n\longrightarrow\infty} x_n = \ell$.

Exercise 7:

$\Longrightarrow$) Suppose that $N =]a,b[$ is a neighbourhood of x_0 and $N \cap A = \{x_1, x_2, \ldots, x_n\}$. By taking $a' = \max\limits_{i=1,\ldots,n}(x_i)$ and $b' = \min\limits_{i=1,\ldots,n}(x_i)$ such that $a < a' < x_0 < b' < b$, we obtain $]a', b'[\cap (A \setminus \{x_0\}) = \emptyset$. Therefore, x_0 is not an accumulation point of A (Proof by contrapositive).

$\Longleftarrow$) I leave it as an exercise (Obvious).

Exercise 8:

$\Longrightarrow$) Let A be a closed subset in $(\mathbb{R}, |.|)$. If $x \in \complement_{\mathbb{R}} A$, then $\complement_{\mathbb{R}} A$ is a neighbourhood of x (see Remark (1.5)). But $\complement_{\mathbb{R}} A \cap A = \emptyset$, which tells us that $x \notin A'$ (Proof by contrapositive).

$\Longleftarrow$) Suppose that A is a subset of $\mathbb{R}$ that contains all its accumulation points. Therefore, no point of $\complement_{\mathbb{R}} A$ is an accumulation point of A. Consequently, for every point of $\complement_{\mathbb{R}} A$, there exists a neighborhood N such that $N \cap A = \emptyset$. Thus, $N \subset \complement_{\mathbb{R}} A$, which shows that $\complement_{\mathbb{R}} A$ is a neighbourhood of each of its points. Therefore, $\complement_{\mathbb{R}} A$ is open, which implies that A is closed.