وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 2: Metric spaces

Exercise 1:

1. Show that the following functions define distances on $\mathbb{R}^n$.

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|, \quad d_\infty(x,y) = \sup_{i=1,\dots,n} (|x_i - y_i|),$$

and draw the open unit balls $B((0,0),1)$ in $\mathbb{R}^2$ for d_1 and d_∞.

2. Show that the following functions are distances on $C[a,b]$.

$$d_1(f,g) = \int_a^b |f(t) - g(t)|dt, d_\infty(f,g) = \sup_{t\in[a,b]} (|f(t) - g(t)|)$$

Exercise 2:

Let $(\mathbb{X}, d)$ be a metric space and f a real increasing function defined on $\mathbb{R}_+$ and satisfying:

$$\begin{cases} f(0) = 0 \\ f(x+y) \leqslant f(x) + f(y), \ \forall x, y \in \mathbb{R}_+. \end{cases}$$

1. Show that the function $d_1 = f \circ d$ is a distance on $\mathbb{X}$.
2. Deduce that the following functions are distances on $\mathbb{X}$.

$$d_2 = \frac{d}{1+d}, \quad d_3 = inf(1, d), \quad d_4 = ln(1+d).$$

Exercise 3:

Let $\mathbb{X}$ be an arbitrary set. Show that:

1. $\delta(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$ is a distance on $\mathbb{X}$.
2. $B(x_0, r) = \begin{cases} \{x_0\} & \text{if } r \leqslant 1 \\ \mathbb{X} & \text{if } r > 1 \end{cases}$

Exercise 4: Let $(\mathbb{X}, d)$ be a metric space and A a subset of $\mathbb{X}$. Show that the following propositions are equivalent:

1. x is an accumulation point of A.
2. Every neighborhood N of x contains an infinite number of points of A.
3. $x \in \mathrm{Cl}(A - \{x\})$.

Exercise 5: Show that a finite intersection of dense open sets of $\mathbb{X}$ is a dense open set in $\mathbb{X}$.

Exercise 6: Let $(\mathbb{X}, d)$ be a metric space and A a subset of $\mathbb{X}$. Show that $Cl(A) = \{x \in \mathbb{X} : d(x, A) = 0\}$.

Exercice 7: Let A and B be two bounded subsets of a metric space $(\mathbb{X}, d)$.

1. Show that $diam(A) = diam(Cl(A))$.
2. Show that $diam(A \cup B) \leq diam(A) + diam(B) + d(A, B)$.

Exercise 8: Let d_1 and d_2 be two distances defined on a set $\mathbb{X}$, such that for every open ball B_{d_1} centered at $x \in \mathbb{X}$, there exists an open ball B_{d_2} also centered at x, such that $B_{d_2} \subset B_{d_1}$. Show that the topology $\mathcal{T}_{d_1}$ induced by d_1 is coarser than the topology $\mathcal{T}_{d_2}$ induced by d_2, i.e., $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$.

Exercise 9: Let d_1 and d_∞ be the two distances defined on $C[a, b]$ (see Exercise 1(2)). Show that the topology $\mathcal{T}_{d_1}$ induced by d_1 is coarser than the topology $\mathcal{T}_{d_\infty}$ induced by d_∞, i.e., $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_\infty}$.