وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
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Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solution of series 2 (Metric spaces)

Exercise 1:

1. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$. Then,

A) ⓒ$_1$

$$\begin{aligned} d_1(x,y) = 0 &\iff \sum_{i=1}^{n} |x_i - y_i| = 0, \\ &\iff |x_i - y_i| = 0, \quad \forall\, 1 \leqslant i \leqslant n, \\ &\iff x_i = y_i, \quad \forall\, 1 \leqslant i \leqslant n, \\ &\iff x = y. \end{aligned}$$

ⓒ$_2$

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{n} |x_i - y_i|, \\ &= \sum_{i=1}^{n} |-(y_i - x_i)|, \\ &= \sum_{i=1}^{n} |y_i - x_i|, \\ &= d_1(y,x). \end{aligned}$$

ⓒ$_3$

$$\begin{aligned} d_1(x,y) &= \sum_{i=1}^{n} |x_i - y_i|, \\ &= \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|, \\ &\leqslant \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i|, \\ &\leqslant d_1(x,z) + d_1(z,y). \end{aligned}$$

From ⓒ$_1$, ⓒ$_2$ and ⓒ$_3$ we conclude that d_1 is a distance on $\mathbb{R}^n$.

B) $\textcircled{c_1}$

$$\begin{aligned} d_\infty(x,y)=0 &\iff \sup_{i=1,\dots,n}(|x_i-y_i|)=0,\\ &\iff |x_i-y_i|=0,\quad \forall\ 1\leqslant i\leqslant n,\\ &\iff x_i=y_i,\quad \forall\ 1\leqslant i\leqslant n,\\ &\iff x=y. \end{aligned}$$

$\textcircled{c_2}$

$$\begin{aligned} d_\infty(x,y) &= \sup_{i=1,\dots,n}(|x_i-y_i|),\\ &= \sup_{i=1,\dots,n}(|-(y_i-x_i)|),\\ &= \sup_{i=1,\dots,n}(|y_i-x_i|),\\ &= d_\infty(y,x). \end{aligned}$$

$\textcircled{c_3}$

$$\begin{aligned} d_\infty(x,y) &= \sup_{i=1,\dots,n}(|x_i-y_i|),\\ &= \sup_{i=1,\dots,n}(|x_i-z_i+z_i-y_i|),\\ &\leqslant \sup_{i=1,\dots,n}(|x_i-z_i|)+\sup_{i=1,\dots,n}|z_i-y_i|,\\ &\leqslant d_\infty(x,z)+d_\infty(z,y). \end{aligned}$$

From $\textcircled{c_1}$, $\textcircled{c_2}$ and $\textcircled{c_3}$ we conclude that d_∞ is a distance on $\mathbb{R}^n$.

2. Let $f,\ g,\ h\in C[a,b]$.

A) $\textcircled{c_1}$

$$\begin{aligned} d_1(f,g)=\int_a^b |f(t)-g(t)|dt,=0 &\iff |f(t)-g(t)|=0\ \ \forall t\in[a,b],\\ &\iff f(t)=g(t)\ \ \forall t\in[a,b],\\ &\iff f\equiv g. \end{aligned}$$

$\textcircled{c_2}$

$$\begin{aligned} d_1(f,g)=\int_a^b |f(t)-g(t)|dt &= \int_a^b |-(g(t)-f(t))|dt,\\ &= \int_a^b |g(t)-f(t)|dt,\\ &= d_1(g,f). \end{aligned}$$

$\textcircled{c_3}$

$$\begin{aligned}
d_1(f,g) &= \int_a^b |f(t)-g(t)|dt,\\
&= \int_a^b |f(t)-h(t)+h(t)-g(t)|dt,\\
&\leqslant \int_a^b |f(t)-h(t)|dt + \int_a^b |h(t)-g(t)|dt,\\
&\leqslant d_1(f,h)+d_1(h,g).
\end{aligned}$$

From $\textcircled{c_1}$, $\textcircled{c_2}$ and $\textcircled{c_3}$ we conclude that d_1 is a distance on $C[a,b]$.

B) $\textcircled{c_1}$

$$\begin{aligned}
d_\infty(f,g) = \sup_{t\in[a,b]} (|f(t)-g(t)|) = 0 &\iff |f(t)-g(t)| = 0 \quad \forall t\in[a,b],\\
&\iff f(t)=g(t) \quad \forall t\in[a,b],\\
&\iff f\equiv g.
\end{aligned}$$

$\textcircled{c_2}$

$$\begin{aligned}
d_\infty(f,g) = \sup_{t\in[a,b]} (|f(t)-g(t)|) &= \sup_{t\in[a,b]} (|-(g(t)-f(t))|),\\
&= \sup_{t\in[a,b]} (|g(t)-f(t)|),\\
&= d_\infty(g,f).
\end{aligned}$$

$\textcircled{c_3}$

$$\begin{aligned}
d_\infty(f,g) &= \sup_{t\in[a,b]} (|f(t)-g(t)|),\\
&= \sup_{t\in[a,b]} (|f(t)-h(t)+h(t)-g(t)|),\\
&\leqslant \sup_{t\in[a,b]} (|f(t)-h(t)|) + \sup_{t\in[a,b]} (|h(t)-g(t)|),\\
&\leqslant d_\infty(f,h)+d_\infty(h,g).
\end{aligned}$$

From $\textcircled{c_1}$, $\textcircled{c_2}$ and $\textcircled{c_3}$ we conclude that d_∞ is a distance on $C[a,b]$.

Exercise 2: Let $(\mathbb{X},d)$ be a metric space and f a real increasing function defined on $\mathbb{R}_+$ and satisfying:

$$\begin{cases} f(0)=0 \\ f(x+y)\leqslant f(x)+f(y),\ \forall x,y\in\mathbb{R}_+. \end{cases} \tag{1}$$

1. $\textcircled{\scriptsize C_1}$ We have $\mathbb{X} \times \mathbb{X} \xrightarrow{d} \mathbb{R}_+ \xrightarrow{f} \mathbb{R}_+$ and for each $x, y \in \mathbb{X}$,

$$\begin{aligned} d_1(x,y) = (f \circ d) = 0 &\iff f\left(d(x,y)\right) = 0, \\ &\iff d(x,y) = 0, \\ &\iff x = y. \end{aligned}$$

$\textcircled{\scriptsize C_2}$ For each $x, y \in \mathbb{X}$ we have,

$$\begin{aligned} d_1(x,y) = (f \circ d)(x,y) &= f\left(d(x,y)\right), \\ &= f\left(d(y,x)\right), \\ &= (f \circ d)(y,x), \\ &= d_1(y,x). \end{aligned}$$

$\textcircled{\scriptsize C_3}$ For each $x, y, z \in \mathbb{X}$ we have, $d_1(x,y) = (f \circ d)(x,y) = (f(d(x,y)))$. Since $d(x,y) \leqslant d(x,z) + d(z,y)$ and f is an increasing function we obtain:

$$\begin{aligned} d_1(x,y) = f\left(d(x,y)\right) &\leqslant f(d(x,z) + d(z,y)), \\ &\leqslant f(d(x,z)) + f(d(z,y)), (\text{ According to } (1)_2) \\ &\leqslant (f \circ d)(x,z) + (f \circ d(z,y), \\ &\leqslant d_1(x,z) + d_1(z,y). \end{aligned}$$

From $\textcircled{\scriptsize C_1}$, $\textcircled{\scriptsize C_2}$ and $\textcircled{\scriptsize C_3}$ we conclude that d_1 is a distance on $\mathbb{X}$.

2. **i)** For $d_2 = \dfrac{d}{1+d}$, we consider the function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $f(x) = \dfrac{x}{1+x}$. Indeed, f is an increase function because $f'(x) = \dfrac{1}{(1+x)^2} > 0$. Moreover, we have $f(0) = 0$ and,

$$\begin{aligned} f(x+y) = \frac{x+y}{1+x+y} &= \frac{x}{1+x+y} + \frac{y}{1+x+y}, \\ &\leqslant \frac{x}{1+x} + \frac{y}{1+y}, \\ &= f(x) + f(y), \end{aligned}$$

for each $x, y \in \mathbb{X}$. Since, $(f \circ d)(x,y) = f(d(x,y)) = \dfrac{d(x,y)}{1+d(x,y)} = d_2(x,y)$ it follows from the first question that $d_2 = \dfrac{d}{1+d}$ is a distance on $\mathbb{X}$.

ii) For $d_3 = inf(1, d)$, we consider the function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $f(x) = inf(1, x)$. Indeed, f is an increase function. Moreover, we have $f(0) = inf(1, 0) = 0$ and,

$$f(x+y) = inf(1, x+y) \leqslant inf(1, x) + inf(1, y) = f(x) + f(y).$$

for each $x, y \in \mathbb{X}$. Since, $(f \circ d)(x,y) = f(d(x,y)) = inf(1, d(x,y)) = d_3(x,y)$, it follows from the first question that $d_3 = inf(1, d)$ is a distance on $\mathbb{X}$.

Exercise 3:

1. $\delta(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

   $\textcircled{c_1}$ $\forall x, y \in \mathbb{X}, \delta(x,y) = 0 \Longleftrightarrow x = y$ (by definition).

   $\textcircled{c_2}$ $\forall x, y \in \mathbb{X}, \delta(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases} = \begin{cases} 1 \text{ if } y \neq x \\ 0 \text{ if } y = x \end{cases} = \delta(y,x).$

   $\textcircled{c_3}$ $\forall x, y, z \in \mathbb{X}$,

   • If $\delta(x,z) = 1$ or $\delta(z,y) = 1$, then $\delta(x,y) \leqslant \delta(x,z) + \delta(z,y)$ because $\delta(x,y) \leqslant 1$.

   • If $\delta(x,z) = \delta(z,y) = 0$, then $x = y = z$ which implies that $0 = \delta(x,y) \leqslant \delta(x,z) + \delta(z,y) = 0$.

   From $\textcircled{c_1}$, $\textcircled{c_2}$ and $\textcircled{c_3}$ we conclude that δ is a distance on $\mathbb{X}$.

2. By definition we have $B(x_0, r) = \{x \in \mathbb{X} : \delta(x_0, x) < r\}$.

   • If $r > 1$, then $\delta(x_0, x) \leqslant 1 < r$ for each $x \in \mathbb{X}$ which implies that $B(x_0, r) = \mathbb{X}$.

   • If $r \leqslant 1$, then $\delta(x_0, x) < r \leqslant 1$. Hence, $\delta(x_0, x) = 0$, which implies that $x = x_0$, and therefore, $B(x_0, r) = \{x_0\}$.

Exercise 4:

1) $\Longrightarrow$ 2) Suppose that there exists a neighborhood N of x containing only finitely many elements $a_1, a_2, \ldots, a_n$ of A. Let

$$r = \inf\{d(x, a_i) : i = 1, \ldots, n\}.$$

Then, the set $N \cap B\left(x, \frac{r}{2}\right)$ is a neighborhood of x that does not contain any point of A other than x. Therefore, x is not an accumulation point of A.

2) $\Longrightarrow$ 3) Suppose that every neighborhood N of x contains infinitely many points of A. Then, $(N \setminus \{x\}) \cap A \neq \emptyset$. Thus, $N \cap (A \setminus \{x\}) \neq \emptyset$, which implies that $x \in Cl(A \setminus \{x\})$.

3) $\Longrightarrow$ 1) Suppose that $x \in Cl\,(A \setminus \{x\})$. Thus, for every neighborhood N of x, we have $N \cap (A \setminus \{x\}) \neq \emptyset$, which implies that $(N \setminus \{x\}) \cap A \neq \emptyset$. Therefore, $x \in A'$ (the derived set of A).

Exercise 5: It suffices to show this for two open sets. Let D_1 and D_2 be two dense open sets in $\mathbb{X}$. On the one hand, it is clear that $D = D_1 \cap D_2$ is an open set in $\mathbb{X}$. On the other hand, since D_1 is dense in $\mathbb{X}$, for every $x \in \mathbb{X}$ and $r > 0$, we have $A = B(x, r) \cap D_1 \neq \emptyset$. Notice that A is a non-empty open set in $\mathbb{X}$, which implies that $A \cap D_2 \neq \emptyset$ because D_2 is dense in $\mathbb{X}$. However, $A \cap D_2 = B(x, r) \cap (D_1 \cap D_2) \neq \emptyset$, which shows that $D = D_1 \cap D_2$ is a dense open set in $\mathbb{X}$.

Exercise 6: Let $B = \{x \in \mathbb{X} : d(x, A) = 0\}$.

• $Cl(A) \overset{?}{\subseteq} \{x \in \mathbb{X} : d(x, A) = 0\} = B$.

Suppose $a \notin B$. Then $d(a, A) = r > 0$, which implies that the open ball $B\left(a, \dfrac{r}{2}\right)$ contains no points of A. In other words, $B\left(a, \frac{r}{2}\right) \cap A = \emptyset$, and consequently, $a \notin \mathrm{Cl}(A)$.

• $B = \{x \in \mathbb{X} : d(x, A) = 0\} \overset{?}{\subseteq} Cl(A)$.

Let $a \notin \mathrm{Cl}(A)$. Then there exists $r > 0$ such that $B(a, r) \cap A = \emptyset$, from which we obtain $d(a, b) \geq r$ for all $b \in A$. This shows that $d(a, A) = \inf\limits_{b \in A} d(a, b) \geq r > 0$, and therefore $d(a, A) \neq 0$; that is, $a \notin B$.

Finally, we conclude that $Cl(A) = \{x \in \mathbb{X} : d(x, A) = 0\}$.

Exercise 7:

1. • $\operatorname{diam}(A) \overset{?}{\leqslant} \operatorname{diam}(\operatorname{Cl}(A))$.

   We have $A \subseteq \operatorname{Cl}(A)$, which implies that

   $$\operatorname{diam}(A) \leqslant \operatorname{diam}(\operatorname{Cl}(A)). \tag{1}$$

   • $\operatorname{diam}(\operatorname{Cl}(A)) \overset{?}{\leqslant} \operatorname{diam}(A)$.

   If $x, y \in \operatorname{Cl}(A)$, then there exist two sequences (x_n) and (y_n) of elements in A such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$. Therefore, there exist $x', y' \in A$ such that, for any $\varepsilon > 0$, we have $d(x, x') < \frac{\varepsilon}{2}$ and $d(y, y') < \frac{\varepsilon}{2}$. Hence,

   $$d(x, y) \leqslant d(x, x') + d(x', y') + d(y', y) < d(x', y') + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = d(x', y') + \varepsilon.$$

   Thus,

   $$\operatorname{diam}(\operatorname{Cl}(A)) = \sup_{x,y \in \operatorname{Cl}(A)} d(x, y) < \sup_{x',y' \in A} d(x', y') + \varepsilon = \operatorname{diam}(A) + \varepsilon,$$

   for all $\varepsilon > 0$, which implies that

   $$\operatorname{diam}(\operatorname{Cl}(A)) \leqslant \operatorname{diam}(A). \tag{2}$$

   From (1) and (2), we conclude that

   $$\operatorname{diam}(\operatorname{Cl}(A)) = \operatorname{diam}(A).$$

2. • $\operatorname{diam}(A \cup B) \overset{?}{\leqslant} \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.

   Suppose $x, y \in A \cup B$, $a \in A$ and $b \in B$, then

   $$\begin{aligned} d(x, y) &\leqslant d(x, a) + d(a, b) + d(b, y), \\ &\leqslant \operatorname{diam}(A) + d(A, B) + \operatorname{diam}(B), \end{aligned}$$

   which implies that

   $$\sup_{x,y \in A \cup B} d(x, y) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B).$$

   Hence, $\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.

Exercise 8: Let $O \in \mathcal{T}_{d_1}$. We want to show that O is also open with respect to d_2. Let $x \in O$. Since O is open for d_1, there exists an open ball $B_{d_1}(x, r_1)$ such that $x \in B_{d_1}(x, r_1) \subseteq O$. By hypothesis, there exists an open ball $B_{d_2}(x, r_2)$ such that $x \in B_{d_2}(x, r_2) \subseteq B_{d_1}(x, r_1) \subseteq O$. Consequently, $O = \underset{x \in O}{\cup} B_{d_2}(x, r_2)$. Thus, O is open for d_2 because it is an union of open balls for d_2. Therefore, $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$.

Exercice 9: Let $B_{d_1}(f, r)$ be an open ball such that $f \in C[a, b]$ and $\varepsilon = \dfrac{r}{b-a}$. Then, according to the previous exercise, it suffices to show that $B_{d_\infty}(f, \varepsilon) \subseteq B_{d_1}(f, r)$. Let $g \in B_{d_\infty}(f, \varepsilon)$, so we have:

$$\sup_{t\in[a,b]} |f(t) - g(t)| < \varepsilon = \frac{r}{b-a}.$$

Thus,

$$d_1(f, g) = \int_a^b |f(t) - g(t)| \, dt \leq \int_a^b \sup_{t\in[a,b]} |f(t) - g(t)| \, dt < \frac{r}{b-a} \int_a^b dt = r.$$

Therefore, $g \in B_{d_1}(f, r)$. Consequently, $\mathcal{T}_{d_1}(f, \varepsilon) \subseteq \mathcal{T}_{d_\infty}(f, r)$.