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Exercises with Solutions

Exercise 1. Consider the following Cauchy problem :

$$\begin{cases} y'(t) & = 2t - y(t) \Big| t \in [0,1] \\ y(0) & = 1. \end{cases} \tag{P}$$

a- Show that the problem (P) admits a unique solution.

b- Verify that the problem (P) admits the equation (1) as a particular solution.

$$y(t) = 2t - 2 + 3e^{-t}. \tag{1}$$

c- Provide the iterative scheme of the fourth-order Runge–Kutta algorithm to solve the problem (P).

d- Apply the fourth-order Runge–Kutta algorithm to this problem with $h = 0.1$ to evaluate the solution at $t = 0.3$. Compare the obtained solution with the exact solution.

Solution .

a- We have $\frac{\delta f}{\delta y} = 1$, which is a continuous and bounded function, so this problem admits a unique solution.

b- We have, according to (1),

$$\begin{aligned} y'(t) =& 2 - 3e^{-t} \\ =& 2 - 3e^{-t} - 2t + 2t \\ =& -y(t) + 2t. \end{aligned}$$

On the other hand, we have $y(0) = -2 + 3 = 1$, from which we deduce that the equation (1) is a particular solution.

c-

$$(RK_4) \begin{cases} y_0 = y(t_0), t_0 = a \text{ et } h = \frac{b-a}{n} \\ y_{i+1} = y_i + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \ldots, n-1 \\ K_1 = f(t_i, y_i) \\ K_2 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_1) \\ K_3 = f(t_i + \frac{h}{2}, y_i + \frac{h}{2}K_2) \\ K_4 = f(t_i + h, y_i + hK_3) \end{cases}$$

d- Apply the fourth-order Runge-Kutta method algorithm RK_4 with $h = 0.1$:

$$(RK_4) \begin{cases} y_0 = y(0) = 1, h = 0.1 \\ y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \ldots, n-1 \\ K_1 = f(t_0, y_0) = f(0, 1) - 1 \\ K_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1) = f(0.05, 1.05) = -0.95 \\ K_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2) = f(0.05, 0.955) = -0.852 \\ K_4 = f(t_0 + h, y_0 + hK_3) = f(0.05, 0.914) = -0.814 \\ y_1 = 1 + \frac{0.1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.943 \end{cases}$$

hence $y(0.1) \simeq y_1 = 0.943$.

$$(RK_4) \begin{cases} y_2 = y_1 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \ldots, n-1 \\ K_1 = f(t_1, y_1) = f(0.1, 0.9430) = -0.743 \\ K_2 = f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_1) = f(0.15, 0.905) = -0.605 \\ K_3 = f(t_1 + \frac{h}{2}, y_1 + \frac{h}{2}K_2) = f(0.15, 0.9127) = -0.6127 \\ K_4 = f(t_1 + h, y_1 + hK_3) = f(0.2, 0.8818) = -0.4818 \\ y_1 = 0.943 + \frac{0.1}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 0.882 \end{cases}$$

then $y(0.2) \simeq y_2 = 0.882$.
By repeating the same process, we obtain : $y(0.3) \simeq y_3 = 0.8436$
Comparison : We have the value of $y(0.3) = 2 \times 0.3 - 2 + 3e^{-0.3} = 0.8225$, then

$$e_i = |0.8225 - 0.8436| = 0.0216$$

Exercise 2.

a- Provide the iterative scheme of the Euler algorithm to solve the problem (P) from exercise 1.

b- Apply the Euler algorithm to this problem with $h = 0.1$ to evaluate the solution at $t = 0.3$. Compare the obtained solution with the exact solution.

Solution .

a-

$$\begin{cases} y_0 = y(t_0), t_0 = a \\ y_{i+1} = y_i + hf(t_i, y_i), i = 1, \dots, n-1 \end{cases}$$

with $h = \frac{b-a}{n}$, et $t_{i+1} = t_i + h$.

b-

$$\begin{aligned} y(0.1) &\simeq y_1 = y_0 + 0.1(2t_0 - y(t_0)) = 1 + 0.1(2 \times 0 - y(0)) = 0.92. \\ y(0.2) &\simeq y_2 = y_1 + 0.1(2t_1 - y(t_1)) = 0.868 \\ y(0.3) &\simeq y_3 = y_2 + 0.1(2t_2 - y(t_2)) = 0.8412 \end{aligned}$$

Then $y(0.3) \simeq y_3 = 0.8412$.
Comparison : The exact value at $t = 0.3$ is $y(0.3) = 0.8225$, so the error made when applying the Euler algorithm is

$$e_i = |0.8225 - 0.8412| = 0.019.$$

The theoretical error is given by

$$e_t \leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h,$$

where $M_2 = \max_{t \in [0,1]} |y''(t)|$ and L is the Lipschitz constant of f with respect to y, which is equal to 1. In addition, we have

$$y''(t) = 3e^{-t}$$

Then $M_2 = \max\limits_{t \in [0,1]} |3e^{-t}| = 3$. Hence ,

$$\begin{aligned} e_t &\leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h \\ &\leq (e^{1(0.3-0)} - 1)\frac{3 \times 0.1}{2 \times 1} \\ &\leq 0.05247 \end{aligned}$$

It is clear that $e_i \leq e_t$, so the Euler method provides a good approximation of the solution to this Cauchy problem at $t = 1$.

Exercice 3. Consider the following differential equation :

$$\begin{cases} y'(t) &= y(t) + t \Big| t \in [0,1] \\ y(0) &= 1. \end{cases}$$

The exact solution of this equation is $y(t) = -1 - t + 2e^t$.
- Numerically approximate the solution of this equation at $t = 1$ using the Euler method by subdividing the interval into 10 equal parts.
- Compare the obtained solution with the exact solution.

Solution .

Let $f(t,y) = y(t) + t$, the points t_i to evaluate for $h = 0.1$ are $t_1 = 0.1, t_2 = 0.2, t_3 = 0.3, \ldots, t_{10} = 1$. By following the same procedure as in the previous examples, we obtain :

$$\begin{aligned}
y(0.1) &\simeq y_1 = y_0 + 0.1 \times f(t_0, y_0) = 1.1 \\
y(0.2) &\simeq y_2 = y_1 + 0.1 \times f(t_1, y_1) = 1.22 \\
y(0.3) &\simeq y_3 = y_2 + 0.1 \times f(t_2, y_2) = 1.362 \\
&\vdots \\
y(1) &\simeq y_{10} = y_9 + 0.1 \times f(t_9, y_9) = 3.1874\,.
\end{aligned}$$

That is, the approximation at $t = 1$ of $y(t)$ is $y_{10} = 3.1874$.

- Comparison of results :

The exact value at $t = 1$ is $y(1) = -1 - 1 + 2e^1 = 3.4366$. Thus, the error actually made when applying the Euler method is

$$e_i = |3.4366 - 3.1874| = 0.25.$$

Now, let's find the theoretical error, which is given by

$$e_t \leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h$$

where $M_2 = \max\limits_{t\in[0,1]} |y''(t)|$ and L is the Lipschitz constant of f with respect to y.

We have,

$$|f(t,y_1) - f(t,y_2)| = |y_1 - y_2| \Rightarrow L = 1.$$

Furthermore, we have,

$$\begin{aligned}
y''(t) &= y'(y) + 1 = y(t) + t + 1 \\
&= 1 + t + (-1 - t + 2e^t) \\
&= 2e^t
\end{aligned}$$

$M_2 = \max\limits_{t\in[0,1]} |2e^t| = 2e$. Hence

$$\begin{aligned}
e_t &\leq (e^{L(b-a)} - 1)\frac{M_2}{2L}h \\
&\leq (e^{1(1-0)} - 1)\frac{2e \times 0.1}{2 \times 1} \\
&\leq 0.4673\,.
\end{aligned}$$

It is clear that $e_i \leq e_t$, so the Euler method provides a good approximation of the solution to this Cauchy problem at $t = 1$.

Exercise 4. Solve the following Cauchy problem using the fourth-order Runge-Kutta method with a step size of $h = 0.1$.

$$\begin{cases} y'(t) = y(t) - t + 2, t \in [0,1] \\ y(0) = 2. \end{cases}$$

Solution .

Apply the fourth-order Runge-Kutta method algorithm RK_4 with $h = 0.1$.

$$(RK_4)\begin{cases}
y_0 = y(0) = 2, h = 0.1 \\
y_1 = y_0 + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4), i = 1, \ldots, n-1 \\
K_1 = f(t_0, y_0) = 0.4 \\
K_2 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_1) = 0.4150 \\
K_3 = f(t_0 + \frac{h}{2}, y_0 + \frac{h}{2}K_2) = 0.4157 \\
K_4 = f(t_0 + h, y_i + hK_3) = 0.4365 \\
y_1 = 1 + \frac{0.25}{6}(K_1 + 2K_2 + 2K_3 + K_4) = 2.4163
\end{cases}$$

By repeating the same process for the other iterations, we obtain the results in the following table :

i	0	1	2	3	4	5	6	7	8	9	10
t_i	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_i	2	2.4163	2.8659	3.5323	3.8793	4.4513	5.0728	5.7492	6.4863	7.2903	8.1684

Exercice 5. Let the following differential equation :

$$\begin{cases} y'(t) & = t - \ln y \\ y(2) & = 3.4 \end{cases}$$

- Calculate $y(2.8)$ using the fourth-order Runge-Kutta method with $h = 0.8$ and then with $h = 0.4$.

Abbreviated solution : .

- $y(2.8)$ with $h = 0.8$ is $y(2.8) \simeq y_1 = 4.255952$.
- $y(2.8)$ with $h = 0.4$ is $y(2.8) \simeq y_2 = 4.255888$.

Exercise 6. Let the following differential equation :

$$y'(t) = \frac{y^2}{t}$$
$$y(1) = 1$$

- Calculate $y(1.5)$ using the fourth-order Runge-Kutta method with a step size of $h = 0.5$.
- Recalculate $y(1.5)$ with $h = 0.25$.

Abbreviated solution : .

- $y(1.5)$ with $h = 0.5$ is $y(1.5) \simeq y_1 = 1.67985$
- $y(1.5)$ with $h = 0.25$ is $y(1.5) \simeq y_2 = 1.68178$

Exercise 7. Let the following Cauchy problem :

$$y'(t) = -y + t + 1 \Big| t \in [0, 1]$$
$$y(0) = 1.$$

a- Calculate the approximation of $y(0.2)$ using the Euler method, with a step size of $h = 0.1$.

b- Calculate the approximation of $y(0.2)$ using the improved Euler method, with a step size of $h = 0.1$.

c- For each method, calculate the error made by comparing the result obtained with the exact solution $y^*(0.2) = 1.018731$.

Abbreviated solution : .

a- $y(0.2) \simeq y_2 = 1.01$ and $|y_2 - y(0.2)| = 0,008731$.

b- $y(0.2) \simeq y_2 = 1.019025$ and $|y_2 - y(0.2)| = 0,000294$.