وزارة التعليــم العالــي والبحــث العلمــي

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Faculty of Sciences
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INTRODUCTION TO METRIC & TOPOLOGICAL SPACES

FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS

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CONTENTS

CHAPTER 1

METRIC SPACES

1.1 Metric spaces

Metric spaces are a fertile field for examples that we will use to study topological spaces and their properties.

The notion of metric space is was introduced in 1906 by Maurice Fréchet and developed and named by Felix Hausdorff in 1914.

Definition 1.1. *Let $\mathbb{X}$ be a non-empty set and $d : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{R}_+$ a real valued function such that for all $x, y, z \in \mathbb{X}$ the following holds:*

C_1) $d(x,y) = 0 \Longleftrightarrow x = y$,

C_2) *d(x,y)=d(y,x); (symmetry).*

C_3) $d(x,y) \leq d(x,z) + d(z,y)$; *(triangle inequality).*

Then d is said to be a metric(or distance) on $\mathbb{X}$, the pair $(\mathbb{X}, d)$ is called a metric space and $d(x,y)$ is referred to as the distance between x and y.

Remark 1.1. *A metric space $(\mathbb{X}, d)$ is a set $\mathbb{X}$ endowed with a metric d. When there is no possibility of confusion, we abbreviate by saying that $\mathbb{X}$ is a metric space.*

Example 1.1. *On $\mathbb{R}^n$ we have the following metrics :*

1) $d_1(x,y) = \sum_{i=1}^{n}|x_i - y_i|$, ***2)*** $d_2(x,y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{\frac{1}{2}}$,

3) $d_\infty(x,y) = \max_{i=1,\dots,n}(|x_i - y_i|)$, ***4)*** $d_p(x,y) = \left(\sum_{i=1}^{n}|x_i - y_i|^p\right)^{\frac{1}{p}}$, $p \geqslant 1$

The metric d_1 is called ℓ^1 metric, d_2 is called the the euclidean metric (or ℓ^2 metric), d_∞ is called the maximum metric(or ℓ^∞ metric) and d_p is called ℓ^p metric.

Example 1.2. *On $C([a,b],\mathbb{R})$(the set of continuous functions from [a,b] to $\mathbb{R}$) we have the following metrics :*

1) $d_1(f,g) = \int_a^b |f(t)-g(t)|dt,$ ***2)*** $d_2(f,g) = [\int_a^b (f(t)-g(t))^2 dt]^{\frac{1}{2}},$

3) $d_\infty(f,g) = \sup_{t\in[a,b]} (|f(t)-g(t)|),$ ***4)*** $d_p(f,g) = [\int_a^b |f(t)-g(t)|^p dt]^{\frac{1}{p}},\ p \geq 1.$

Example 1.3. *The function $d_u : \mathbb{R} \longrightarrow \mathbb{R}_+$ given by $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$ and is called usual metric(or euclidean metric) on $\mathbb{R}$.*

Example 1.4. *Let $\mathbb{X}$ be a non-empty set and $d : \mathbb{X}\times\mathbb{X} \longrightarrow \mathbb{R}$ the function defined by*

$$\delta(x,y) = \begin{cases} 1 \text{ if } x \neq y, \\ 0 \text{ if } x = y. \end{cases} \tag{1.1}$$

Then, d is a metric on $\mathbb{X}$ and is called the discrete metric.

Proposition 1.1. *Let $(\mathbb{X},d)$ be a metric space. Then,*

$$|d(x,z)-d(y,z)| \leqslant d(x,y), \tag{1.2}$$

for all $x,y,z \in \mathbb{X}$

Proof. *Using the triangle inequality for metrics we obtain*

$$d(x,z)-d(y,z) \leqslant d(x,y). \tag{i}$$

Again, by using the triangle inequality for metrics we can see that

$$-d(x,y) \leqslant d(x,z)-d(y,z). \tag{ii}$$

*Then, the inequality (1.2) follows from (**i**) and (**ii**).*

Definition 1.2. *Let $(\mathbb{X},d_\mathbb{X})$ be a metric space and let A be a subset of $\mathbb{X}$. We define a metric $d_A : A\times A \longrightarrow \mathbb{R}_+$ on A by $d_A(x,y) = d_\mathbb{X}(x,y)$ for all $x,y \in A$. Then, (A,d_A) is a metric space, which is said to be a subspace of $(\mathbb{X},d_\mathbb{X})$.*

Remark 1.2. *The metric d_A is just the function $d_{\mathbb{X}}$ restricted to the subset $A \times A$ of $\mathbb{X} \times \mathbb{X}$.*

1.2 Open balls, closed balls and spheres

Definition 1.3. *Let $(\mathbb{X}, d)$ be a metric space. Let $a \in \mathbb{X}$ and r any positive real number. Then,*

1) *the open ball around a of radius r is defined as follows:*

$$B(a,r) = \{x \in \mathbb{X} \ / \ d(a,x) < r\}.$$

2) *the closed ball around a of radius r is defined as follows:*

$$B_f(a,r) = \{x \in \mathbb{X} \ / \ d(a,x) \leq r\}.$$

3) *the sphere centred at a of radius r is defined as follows:*

$$S(a,r) = \{x \in \mathbb{X} \ / \ d(a,x) = r\}.$$

Remark 1.3. *$B_f(a,r) = B(a,r) \cup S(a,r)$ for all $a \in \mathbb{X}$ and $r > 0$.*

Example 1.5. *In $\mathbb{R}$ with the euclidean metric d_u we have:*

- $B(a,r) = \{x \in \mathbb{R} \ / \ |x-a| < r\} = (a-r, a+r)$.
- $B_f(a,r) = \{x \in \mathbb{R} \ / \ |x-a| \leq r\} = [a-r, a+r]$.
- $S(a,r) = \{x \in \mathbb{R} \ / \ |x-a| = r\} = \{a-r, a+r\}$.

Example 1.6. *In $\mathbb{R}^2$ with the euclidean metric d_2 we have:*

- *$B(a,r)$ is the open disc centred at $a = (a_1, a_2) \in \mathbb{R}^2$ of radius r.*
- *$B_f(a,r)$ is the closed disc centred at $a = (a_1, a_2) \in \mathbb{R}^2$ of radius r.*
- *$S(a,r)$ is the circle centred at $a = (a_1, a_2) \in \mathbb{R}^2$ of radius r.*

Example 1.7. *In $\mathbb{R}^2$, if we take $a = (0,0) \in \mathbb{R}^2$ and $r = 1$ we obtain:*

- $B_{d_1}((0,0),1) = \left\{(x_1,x_2) \in \mathbb{R}^2 \ / \ d_1((x_1,x_2),(0,0)) = |x_1| + |x_2| < 1\right\}$,
- $B_{d_2}((0,0),1) = \left\{(x_1,x_2) \in \mathbb{R}^2 \ / \ d_2((x_1,x_2),(0,0)) = \sqrt{(x_1)^2 + (x_2)^2} < 1\right\}$,
- $B_{d_\infty}((0,0),1) = \left\{(x_1,x_2) \in \mathbb{R}^2 \ / \ d_\infty((x_1,x_2),(0,0)) = max(|x_1|,|x_2|) < 1\right\}$.

Hence, the unit ball (open ball) $B((0,0),1)$ takes the following forms:

Example 1.8. *In $\mathbb{R}^3$ equipped with the euclidean metric d_2 we have:*

- *$S(a,r)$ is the sphere centred at a of radius r.*

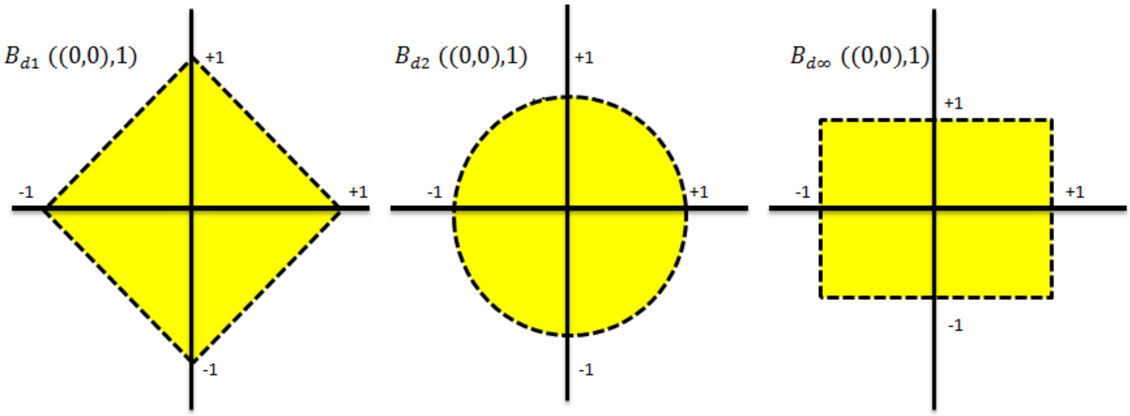


Figure 1.1: Open ball $B((0,0),1)$ in $\mathbb{R}^2$ with d_1, d_2 and d_∞

- *$B(a,r)$ is the open ball (excluding the boundary points that constitute the sphere) centred at a of radius r.*
- *$B_f(a,r)$ is the closed ball (including the boundary points that constitute the sphere) centred at a of radius r.*

Example 1.9. *In the discrete metric $(\mathbb{X},\delta)$ (see Example (1.4)), we have:*

$$B(a,r) = \begin{cases} \{a\} & \text{if } r \leq 1, \\ \mathbb{X} & \text{if } r > 1, \end{cases}$$

for all $a \in \mathbb{X}$ and $r > 0$.

1.3 Open sets, closed sets and neighbourhood

Definition 1.4. *Let $(\mathbb{X},d)$ be a metric space. A set $O \subset \mathbb{X}$ is called open if every point $x \in O$ is the centre of an open ball contained in O. That is,*

$$\forall x \in O, \exists r > 0 \text{ such that } B(x,r) \subseteq O.$$

Definition 1.5. *Let $(\mathbb{X},d)$ be a metric space. A subset F of $\mathbb{X}$ is said to be closed in $(\mathbb{X},d)$ if its complement, $C_{\mathbb{X}}F\,(\text{or}\,F^C)$, is open in $(\mathbb{X},d)$.*

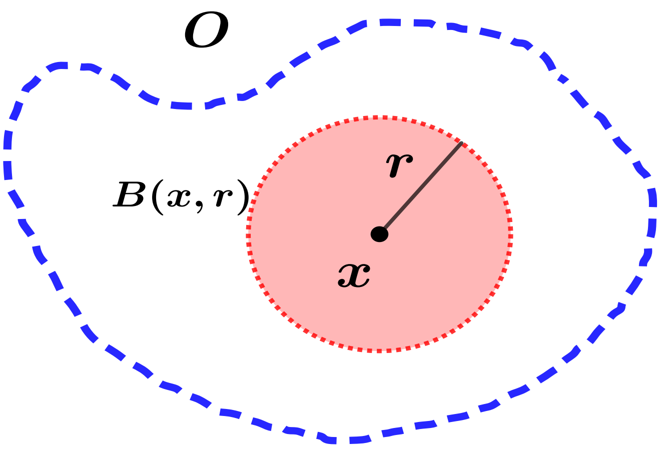
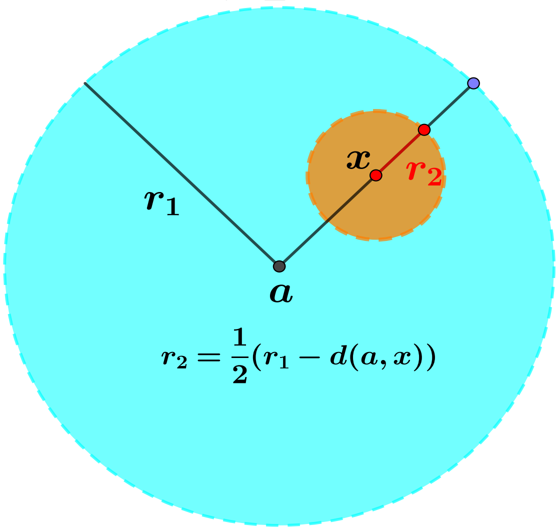


Figure 1.2: Open set **O**

Proposition 1.2. *Let $(\mathbb{X}, D)$ be a metric space. Let $a \in \mathbb{X}$ and $r > 0$. Then the open ball $B(a,r)$ is an open set.*

Proof. *Let $B(a,r_1)$ be an open ball in $(\mathbb{X},d)$. Then, for all $x \in B(a,r_1)$ we have $d(a,x) < r_1$. By taking $r_2 = \frac{1}{2}(r_1 - d(a,x))$ we obtain $B(x,r_2) \subset B(a,r_1)$.*



Remark **1.4.** *Using the previous preposition we conclude that the closed ball $B_f(a,r)$ is a closed set.*

Proposition 1.3. *Let $(\mathbb{X},d)$ be a metric space. Any open set in $\mathbb{X}$ is an union of open balls.*

Proof. *Let O be an open subset of $\mathbb{X}$. Then, for all $x \in O$ there exists $r > 0$ such that $B(x,r) \subseteq O$ which implies that $O = \bigcup_{x\in O} \{x\} \subseteq \bigcup_{x\in O} B(x,r) \subseteq O$. Hence, $O = \bigcup_{x\in O} B(x,r)$.*

Definition 1.6. *Let $(\mathbb{X},d)$ be a metric space and let $x \in \mathbb{X}$. A subset $\mathcal{N}$ of $\mathbb{X}$ is said to be neighbourhood of x in $(\mathbb{X},d)$ if there is an $r > 0$ such that $B(x,r) \subseteq \mathcal{N}$, that is, if $\mathcal{N}$ contains an open ball centred at x with radius r. We denote by $\mathcal{N}(x)$ the set of neighbourhoods of x.*

Example 1.10. ***1)*** *In $\mathbb{R}$ with the Euclidean metric, the set $\mathbb{R}_+$ (the positive real numbers) is a neighbourhood of $x = 2$ because the open ball $B(2,0.5)$ is completely contained in $\mathbb{R}_+$.*

2) *In $\mathbb{R}$ with the Euclidean metric, the set $\mathcal{Z}$ is not a neighbourhood of $x = 2$ because any open ball centred at $x = 2$ will contain some non-integers.*

Remark 1.5. *Using proposition (1.3) we can easily prove that a set is open if and only if it is a neighbourhood of each of its points (Do yourself!).*

Proposition 1.4. *Let $(\mathbb{X},d)$ be a metric space. Then, the sphere $S(x,r)$ is a closed set in $(\mathbb{X},d)$.*

Proof. *The complement of the sphere $S(x,r)$ is the union $\complement_{\mathbb{X}} B_f(x,r) \bigcup B(x,r)$ which is an open set in $\mathbb{X}$ because it is the union of two open sets in $\mathbb{X}$. So the sphere $S(x,r)$ is a closed set in $(\mathbb{X},d)$.*

Proposition 1.5. *Let $(\mathbb{X},d)$ be a metric space. Then, the open sets in $\mathbb{X}$ satisfy the following properties.*

1) *$\mathbb{X}$ is open and $\emptyset$ is open.*

2) *Any union of open sets in $\mathbb{X}$ is an open set in $\mathbb{X}$.*

3) *Any finite intersection of open sets in $\mathbb{X}$ is an open set in $\mathbb{X}$.*

Proof.

1) *Note that it is vacuously true that the empty set contains an open ball about each of its points, since it contains no points. And the set* $\mathbb{X}$ *contains an open ball about each of its points because every open ball is a subset of* $\mathbb{X}$.

2) *Assume that* $\{O_i : i \in I\}$ *is a collection (finite or infinite) of open sets in* $(\mathbb{X}, d)$ *and let* $x \in \bigcup_{i\in I} O_i$. *So, there exists* $i_0 \in I$ *such that* $x \in O_{i_0}$. *Since,* O_{i_0} *is open, there is an* $r > 0$ *such that* $B(x,r) \subseteq O_{i_0}$. *Hence,* $B(x,r) \subseteq O_{i_0} \subseteq \bigcup_{i\in I} O_i$ *which implies that* $\bigcup_{i\in I} O_i$ *is an open set in* $(\mathbb{X}, d)$.

3) *Assume that* $\{O_1, O_2, \ldots, O_n\}$ *is a finite collection of open sets in* $(\mathbb{X}, d)$ *and let* $x \in \bigcap_{i=1}^{n} O_i$. *Then* $x \in O_i$ *for each* $i = 1, 2, \ldots, n$. *So, for each* $i = 1, 2, \ldots, n$, *there is an* $r_i > 0$ *such that* $B(x, r_i) \subseteq O_i$. *Let* $r = min(r_1, r_2, \ldots, r_n)$. *Then* $B(x,r) \subseteq O_i$ *for all* $i = 1, 2, \ldots, n$, *which implies that* $B(x,r) \subseteq \bigcap_{i=1}^{n} O_i$. *Hence,* $\bigcap_{i=1}^{n} O_i$ *is an open set in* $(\mathbb{X}, d)$.

Example 1.11. *In* $\mathbb{R}$ *with the Euclidean metric (usual metric), the set*

$$A = \bigcap_{i=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

is an intersection of open sets but is in fact not open.

By applying De Morgan's laws to the previous proposition, we can easily prove the following similar proposition for closed sets.

Proposition 1.6. *Let* $(\mathbb{X}, d)$ *be a metric space. Then, the closed sets in* $\mathbb{X}$ *satisfy the following properties.*

1. $\mathbb{X}$ *is closed and* $\emptyset$ *is closed.*
2. *Any intersection of closed sets in* $\mathbb{X}$ *is an closed set in* $\mathbb{X}$.
3. *Any finite union of closed sets in* $\mathbb{X}$ *is an closed set in* $\mathbb{X}$.

Proposition 1.7. *Let* $(\mathbb{X}, d)$ *be a metric space and let* $x \in \mathbb{X}$.

1. *Any union of neighborhoods of* x *is also its neighborhood.*
2. *Any finite intersection of neighborhoods of* x *is also its neighborhood.*

Proof.

1) *Suppose that $\{\mathcal{N}_i : i \in I\}$ is a collection (finite or infinite) of neighborhoods of x in $(\mathbb{X},d)$. Then for each $i \in I$, there is an $r_{i>0}$ such that $B(x,r_i) \subseteq \mathcal{N}_i$. Hence $\bigcup_{i\in I} B(x,r_i) \subseteq \bigcup_{i\in I} \mathcal{N}_i$ which implies that there exists $i_0 \in I$ such that $B(x,r_{i_0}) \subseteq \bigcup_{i\in I} \mathcal{N}_i$. So, $\bigcup_{i\in I} \mathcal{N}_i$ is a neighborhood of x in $(\mathbb{X},d)$.*

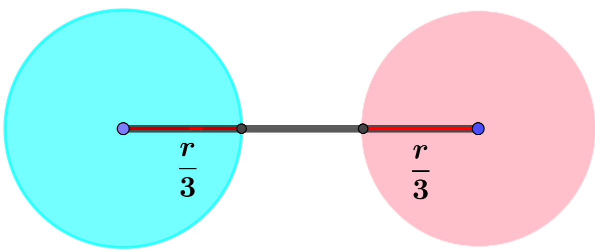
2) *Assume that $\{\mathcal{N}_1,\mathcal{N}_2,...,\mathcal{N}_n\}$ is a finite collection of neighborhoods of x in $(\mathbb{X},d)$. Then, for each $i = 1,2,...,n$, there is an $r_i > 0$ such that $B(x,r_i) \subseteq \mathcal{N}_i$. Let $r = min(r_1,r_2,...,r_n)$. Then $B(x,r) \subseteq \mathcal{N}_i$ for all $i = 1,2,...,n$, which implies that $B(x,r) \subseteq \bigcap_{i=1}^{n} \mathcal{N}_i$. Hence, $\bigcap_{i=1}^{n} \mathcal{N}_i$ is a neighborhood of x in $(\mathbb{X},d)$.*

Definition 1.7. *Let $(\mathbb{X},d)$ be a metric space. Let $x,y \in \mathbb{X}$. We say that x and y can be separated by neighborhoods if there exists a neighborhood $U \in \mathcal{N}(x)$ and a neighborhood $V \in \mathcal{N}(y)$ such that U and V are disjoint, i.e., $U \bigcap V = \emptyset$.*

Definition 1.8. *A metric space $((\mathbb{X},d))$ is said to be a Hausdorff space if every two distinct points of $\mathbb{X}$ have disjoint neighborhoods.*

Proposition 1.8. *Any metric space is a Hausdorff space.*

Proof. *Let $(\mathbb{X},d)$ be a metric space and let $x,y \in \mathbb{X}$ such that $x \neq y$. So there exists $r > 0$ such that $d(x,y) = r$. Hence, if we take $U = B(x,\frac{r}{3})$ and $V = B(y,\frac{r}{3})$ we obtain $U \bigcap V = \emptyset$, which shows that $(\mathbb{X},d)$ is a* ***Hausdorff*** *space.*



1.4 Interior, exterior, boundary and closure

Definition 1.9. *Let* $(\mathbb{X},d)$ *be a metric space and let* $A \subseteq \mathbb{X}$. *A point* $x \in A$ *is called an interior point (or an inner point,) of* A*, if and only if there exists* $r > 0$ *such that* $B(x,r) \subseteq A$. *The set of all interior points of* A *is called the interior of* A *and is denoted by* $Int(A)$ *(or* $\mathring{A}$*).*

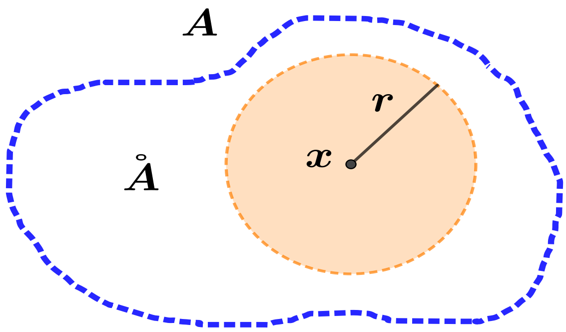


Figure 1.3: *Interior point*

Example 1.12. *In* $\mathbb{R}$ *with the Euclidean metric, we have* $Int([0,1]) = (0,1)$ *(because for any point other than* 0 *or* 1*, we can fit a ball inside* $[0,1]$*).*

Example 1.13. *In* $\mathbb{R}$ *with the Euclidean metric, we have* $Int(\mathbb{Q}) = \emptyset$ *(because for any* $x \in \mathbb{Q}$*, there is no ball* $(x-r, x+r)$ *that lies entirely within* $\mathbb{Q}$*).*

Example 1.14. *In* $\mathbb{R}$ *with the Euclidean metric, we have* $Int((0,1)) =]0,1[$ *(because for all* $x \in (0,1)$*, there is* $r > 0$ *such that* $(x-r, x+r) \subseteq (0,1)$*).*

Proposition 1.9. *Let* $A \subset \mathbb{X}$ *where* $(\mathbb{X},d)$ *is a metric space. Then*

1. $Int(A)$ *is open.*
2. $Int(A)$ *is the largest open subset contained in* A*.*

Proof.

1) *If* $x \in Int(A)$ *then there exists* $r > 0$ *such that* $B(x,r) \subset A$. *Since* $B(x,r))$ *is open, it only contains interior points of* A*, thus* $B(x,r) \subset Int(A)$. *Hence* $Int(A)$ *is open per Definition(1.4).*

2) *Fix a set A and let $G \subset A$ be an open set. Let $x \in G$ arbitrary. Since G is open, x is an interior point of G and there exists some $r > 0$ such that $B(x,r) \subset G$. But since $G \subset A$, then $B(x,r) \subset A$ and thus $x \in Int(A)$, showing that $G \subset Int(A)$. Since $Int(A)$ is always open, $Int(A)$ is the largest open subset contained in A.*

Remark 1.6. *From the previous proposition we conclude that if A is open, then $Int(A) = A$ and if $Int(A) = A$ then A is open.*

Definition 1.10. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an exterior point of A, if and only if there exists $r > 0$ such that $B(x,r) \subseteq \complement_{\mathbb{X}} A$. The set of all exterior points of A is called the exterior of A and is denoted by $Ext(A)$.*

Example 1.15. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Ext((0,1)) = (-\infty, 0) \cup (1, +\infty).$$

Definition 1.11. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an adherent point of A, if and only if for every real $r > 0$, we have $B(x,r) \cap A \neq \emptyset$. The set of all adherent points of A is called the closure of A and is denoted by $Cl(A)$ (or $\bar{A}$).*

Example 1.16. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Cl((0,1)) = [0,1].$$

Proposition 1.10. *Let $A \subset \mathbb{X}$ where $(\mathbb{X}, d)$ is a metric space. Then*

$$Cl(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Int(A).$$

Proof. *Let $x \in \mathbb{X}$. Then we have*

$$\begin{aligned} x \in \complement_{\mathbb{X}} Int(A) &\iff x \notin Int(A) \\ &\iff \forall r > 0, B(x,r) \nsubseteq A \\ &\iff \forall r > 0, B(x,r) \subseteq \complement_{\mathbb{X}} A \\ &\iff \forall r > 0, B(x,r) \cap \complement_{\mathbb{X}} A \neq \emptyset \\ &\iff x \in Cl(\complement_{\mathbb{X}} A). \end{aligned}$$

Proposition 1.11. *Let $A \subset \mathbb{X}$ where $(\mathbb{X}, d)$ is a metric space. Then,*

1. *$Cl(A)$ is closed.*
2. *$Cl(A)$ is the smallest closed set containing A.*

Proof. *Do yourself!*

Remark 1.7. *From the previous proposition we conclude that if A is closed, then $Cl(A) = A$ and if $Cl(A) = A$ then A is closed.*

Definition 1.12. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called a boundary point of A, if and only if x is neither an interior point nor an exterior point of A. The set of all boundary points of A is called the boundary of A and is denoted by $\partial(A)$.*

Example 1.17. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$\partial((0,1)) = \{0,1\}.$$

Remark 1.8. *We can define the boundary of A as follows:*

$$\begin{aligned} \partial(A) &= Cl(A) \cap Cl(\complement_{\mathbb{X}} A) (or \overline{A} \cap \overline{\complement_{\mathbb{X}} A}). \\ &= Cl(A) \setminus Int(A) (or \overline{A} \setminus \mathring{A}). \end{aligned}$$

Definition 1.13. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an accumulation point(or a limit point) of A, if and only if for every real $r > 0$, we have $(B(x,r) \setminus \{x\}) \cap A \neq \emptyset$. The set of all accumulation points of A is called the derived set of A and is denoted by A'.*

Example 1.18. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$\mathbb{Z}' = \emptyset, \qquad \mathbb{Q}' = \mathbb{R}.$$

Remark 1.9. *We have $Cl(A) = A \cup A'$ and a set A is closed if $A' \subset A$.*

Definition 1.14. *Let $(\mathbb{X}, d)$ be a metric space and let $A \subseteq \mathbb{X}$. A point $x \in A$ is called an isolate point in A, if and only if there exists a real $r > 0$ such that $(B(x,r) \setminus \{x\}) \cap A = \emptyset$ (or $B(x,r) \cap A = \{x\}$). The set of all isolate points in A is denoted by $Is(A)$.*

Remark 1.10. *If $x \in A$ is not an accumulation point, it is called isolated in A.*

Example 1.19. *In $\mathbb{R}$ with the Euclidean metric (usual metric), we have*

$$Is(\mathbb{Z}) = \mathbb{Z}, \qquad Is(\mathbb{Q}) = \emptyset.$$

1.5 Distance between two sets, Diameter

Definition 1.15. *Let $(\mathbb{X}, d)$ be a metric space with $A, B \subset \mathbb{X}$ and $a \in \mathbb{X}$. We define the distance between a point and a set, and the distance between two sets as follows:*

$$d(a, A) = \inf_{x \in A} d(a, x), \qquad d(A, B) = \inf_{x \in A, y \in B} d(x, y), \tag{1.3}$$

Remark 1.11.

1. *From the previous definition we conclude that $d(A, B) = d(B, A)$.*
2. *$d(A, B)$ is not a distance on $\mathcal{P}(\mathbb{X})$ (the power set of $\mathbb{X}$). For example, in $(\mathbb{R}, |.|)$, if we take $A = [-2, 4]$ and $B = [4, 6]$ we obtain $d(A, B) = 0$ but $A \neq B$.*
3. *$\forall A, B \subset \mathbb{X}$, $A \cap B \neq \emptyset \Longrightarrow d(A, B) = 0$. The reciprocal of the previous implication is not true. For example, in $(\mathbb{R}, |.|)$, if we take $A = \left\{\frac{n+1}{n}, n \in \mathbb{N}^*\right\}$ and $B = \{1\}$ we obtain $d(A, B) = \inf_{n \in \mathbb{N}^*} \left|1 - \frac{n+1}{n}\right| = 0$ but $A \cap B = \emptyset$.*

Proposition 1.12. *Let $(\mathbb{X}, d)$ be a metric space, let A be a subset of $\mathbb{X}$ and let $x \in \mathbb{X}$. Then, we have*

1) $x \in Cl(A) \iff d(x, A) = 0.$

2) $x \in Ext(A) \iff d(x, A) > 0.$

Proof.

1. $\Longrightarrow$) $x \in Cl(A) \Longrightarrow \forall \varepsilon > 0,\ B(x,\varepsilon) \cap A \neq \emptyset \Longrightarrow \forall \varepsilon > 0,\ d(x,A) < \varepsilon \Longrightarrow d(x,A) = 0.$
   $\Longleftarrow$) *Suppose that* $x \notin Cl(A)$, *then there exists* $r > 0$ *such that* $B(x,r) \cap A = \emptyset$ *(By negation). Hence*
   $$\forall y \in A,\ d(x,y) \geqslant r,$$
   which shows that
   $$d(x,A) = \inf_{y \in A} f(x,y) \geqslant r > 0.$$
2. *By negation of* ***(1)****(Do yourself!).*

Definition 1.16. *Let* $(\mathbb{X},d)$ *be a metric space and* A *a subset of* $\mathbb{X}$. *Then the set* A *is said to be* *bounded* *if there exists some* $x_0 \in \mathbb{X}$ *and* $r > 0$ *such that* $A \subseteq B(x_0,r)$.

Remark 1.12. *From the previous definition we conclude that the finite subsets of* $\mathbb{X}$ *are bounded.*

Definition 1.17. *Let* A *be a non-empty subset of a metric space* $(\mathbb{X},d)$. *The* *diameter* *of* A *is defined by*

$$diam(A) = \sup_{x,y \in A} d(x,y). \tag{1.4}$$

Proposition 1.13. *A non-empty subset* A *of a metric space* $(\mathbb{X},d)$ *is bounded if and only if* $diam(A) < +\infty$.

Proof.

$\Longrightarrow$) *Suppose* A *is bounded, then there exists* $r > 0$ *such that* $A \subseteq B(x,r)$. *Hence* $diam(A) \leqslant 2r < +\infty$.

$\Longleftarrow$) *Suppose* $diam(A) < +\infty$, *then for every* $x \in A$ *we have* $A \subseteq B(x, diam(A))$ *which implies that* A *is bounded.*

1.6 Equivalent metrics

Definition 1.18. *Let $(\mathbb{X}, d)$ be a metric space. Let $\mathcal{T}_d$ be the collection of subsets U of $\mathbb{X}$ such that for each $x \in U$ there exists $r > 0$ with $B(x,r) \subset U$. Then $(\mathbb{X}, \mathcal{T}_d)$ is called the topological space defined by the metric d and call $\mathcal{T}_d$ the topology on $\mathbb{X}$ defined by d.*

Sometimes different metrics on a set give rise to the same topology.

Definition 1.19. *Let d_1 and d_2 be metrics on a set $\mathbb{X}$. We say that d_1 and d_2 are equivalent if they define the same topology , i.e. if $\mathcal{T}_{d_1} = \mathcal{T}_{d_2}$.*

Proposition 1.14. *Suppose that metrics d_1 and d_2 on $\mathbb{X}$ are such that for some $\kappa > 0$ we have*

$$\frac{1}{\kappa} d_1(x,y) \leqslant d_2(x,y) \leqslant \kappa d_1(x,y),$$

for all $x, y \in \mathbb{X}$. Then d_1 and d_2 are equivalent (or Lipschitz-equivalent).

Proof. *Let $\mathcal{T}_1$ be the topology defined by d_1 (i.e. $\mathcal{T}_1 = \mathcal{T}_{d_1}$) and let $\mathcal{T}_2$ be the topology defined by d_2 (i.e. $\mathcal{T}_2 = \mathcal{T}_{d_2}$). We must show that a subset U of $\mathbb{X}$ belongs to $\mathcal{T}_1$ if and only if it belongs to $\mathcal{T}_2$.*

Suppose U belongs to $\mathcal{T}_1$. Let $x \in U$. Then there exists some $r > 0$ such that $B_{d_1}(x,r) \subset U$, i.e.

$$\{y \in \mathbb{X} \setminus d_1(x,y) < r\} \subset U.$$

Consider $B_{d_2}(x, r/\kappa)$. If $y \in B_{d_2}(x, r/\kappa)$ then $d_2(x,y) < r/\kappa$. But $\frac{1}{\kappa} d_1(x,y) \leqslant d_2(x,y)$ and so, for $y \in B_{d_2}(x, r/\kappa)$ we have $d_1(x,y) < \kappa d_2(x,y) < \kappa . \frac{r}{\kappa} = r$. Hence $y \in B_{d_1}(x,r)$ whenever $y \in B_{d_2}(x, r/\kappa)$. But $B_{d_1}(x,r) \subset U$ and so $B_{d_2}(x, r/\kappa) \subset B_{d_1}(x,r) \subset U$. Thus, for $x \in U$, there exists some $r' > 0$ (namely $r' = r/\kappa$) such $B_{d_2}(x, r') \subset U$. Thus U is open in the topology determined by d_2, i.e. $U \in \mathcal{T}_1$ implies that $U \in \mathcal{T}_2$

Now suppose that $U \in \mathcal{T}_2$. For $x \in U$ there exists some $r > 0$ with $B_{d_2}(x,r) \subset U$. Now if $d_1(y,x) < r/\kappa$ we have

$$d_2(x,y) \leqslant \kappa d_1(x,y) \leqslant \kappa . \frac{r}{\kappa} = r,$$

so $B_{d_1}(x, r/\kappa) \subset B_{d_2}(x,r) \subset U$, and so $U \in \mathcal{T}_1$. Thus $U \in \mathcal{T}_1$ if and only if $U \in \mathcal{T}_2$ and hence $\mathcal{T}_1 = \mathcal{T}_2$.

Proposition 1.15. *The three metrics, d_1, d_2 and d_∞, on $\mathbb{R}^n$ (see Example(1.1)) are equivalent.*

Proof. *Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Note that*

$$\begin{aligned} d_2(x,y) &= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \\ &\leqslant \sqrt{\sum_{i=1}^{n}(d_\infty(x,y))^2} \qquad (\text{since } |x_i - y_i| \leqslant d_\infty(x,y), \quad \forall 1 \leqslant i \leqslant n) \\ &= \sqrt{n}\, d_\infty(x,y) \end{aligned}$$

and so

$$d_2(x,y) \leqslant n d_\infty(x,y). \tag{1.5}$$

Also we have

$$\begin{aligned} d_\infty(x,y) &= \max_{i=1,...,n}(|x_i - y_i|) \\ &= |x_j - y_j| \quad \text{for some } j \\ &= \sqrt{(x_j - y_j)^2} \\ &\leqslant \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_2(x,y), \end{aligned}$$

and so $d_\infty(x,y) \leqslant d_2(x,y)$ and certainly

$$d_\infty(x,y) \leqslant n d_2(x,y). \tag{1.6}$$

Combining (1.5) and (1.6) we get

$$\frac{1}{n} d_2(x,y) \leqslant d_\infty(x,y) \leqslant n d_2(x,y),$$

and so d_2 and d_∞ are equivalent.

Clearly $d_\infty(x,y) \leqslant d_1(x,y)$ and so

$$\frac{1}{n} d_\infty(x,y) \leqslant d_1(x,y). \tag{1.7}$$

Also, $d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ and each $|x_i - y_i| \leqslant d_\infty(x,y)$ so that

$$d_1(x,y) \leqslant n d_\infty(x,y). \tag{1.8}$$

Combining (1.7) and (1.8) we get

$$\frac{1}{n}d_\infty(x,y) \leqslant d_1(x,y) \leqslant n d_\infty(x,y),$$

and so d_1 and d_∞ are equivalent.

We have now shown that d_2 and d_∞ define the same topology and that d_1 and d_∞ define the same topology and hence d_1, d_2 and d_∞ all define the same topology, i.e. d_1, d_2 and d_∞ are equivalent.

1.7 Finite metric products

Let $\{(\mathbb{X}_i, d_i) : i = 1, ..., n\}$ be a collection of metric spaces and let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ be arbitrary points in the product $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$. Define

$$d(x,y) = max\{d_i(x_i, y_i) : 1 \leqslant i \leqslant n\}.$$

Proposition 1.16. $(\mathbb{X}, d)$ *is a metric space.*

Proof. *Clearly $d(x,y) \geqslant 0$ and $d(x,y) = 0$ if and only if $d_i(x_i,y_i) = 0$ for $1 \leqslant i \leqslant n$, which the case if and only if $x_i = y_i$ for $1 \leqslant i \leqslant n$, i.e., if and only if $x = y$. It is equally clear that $d(x,y) = d(y,x)$. It remains to verify the triangle inequality. Observe that*

$$d_i(x_i,z_i) \leqslant d_i(x_i,y_i) + d_i(y_i,z_i),$$

for all $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n) \in \mathbb{X}$. This implies

$$d_k(x_k,z_k) \leqslant max\{d_i(x_i,y_i) : 1 \leqslant i \leqslant n\} + max\{d_i(y_i,z_i) : 1 \leqslant i \leqslant n\},$$

for $k = 1, 2, ..., n$. So

$$d(x,z) = max\{d_k(x_k,z_k) : 1 \leqslant k \leqslant n\} \leqslant d(x,y) + d(y,z).$$

Hence, $(\mathbb{X}, d)$ is a metric space.

Definition 1.20. *The metric space obtained by taking*

$$d(x,y) = max\{d_i(x_i,y_i) : 1 \leqslant i \leqslant n\}, \tag{1.9}$$

as the distance on $\mathbb{X}$, is called the product of the metric spaces $(\mathbb{X}_1, d_1)$, $(\mathbb{X}_2, d_2)$,..., $(\mathbb{X}_n, d_n)$.

Remark 1.13.

i) *The functions*

$$\textbf{1)}\ d_1(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i),$$

$$\textbf{2)}\ d_2(x,y) = \left[\sum_{i=1}^{n} (d_i(x_i, y_i))^2\right]^{\frac{1}{2}},$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, $z = (z_1, z_2, ..., z_n) \in \mathbb{X}$, are also metrics on $\mathbb{X}$. The proof of the statement that d_1 and d_2 are metrics on $\mathbb{X}$ is almost trivial.
ii) *The metrics d, d_1 and d_2 on $\mathbb{X}$ are equivalent. Indeed*

$$d(x,y) \leqslant d_2(x,y) \leqslant d_1(x,y) \leqslant nd(x,y).$$

Proposition 1.17. *The open ball $B(x,r)$, $x = (x_1, x_2, ..., x_n)$ and $r > 0$, in $\mathbb{X}$ is the product of the open balls $B_1(x_1, r)$, $B_2(x_2, r)$,...$B_n(x_n, r)$. That is*

$$B(x,r) = \prod_{i=1}^{n} B_i(x_i, r),$$

where $B_i(x_i, r)$ is the open ball centered in $x_i \in \mathbb{X}_i$ with radius $r > 0$.

Proof. *We have $y \in B(x,r)$ if and only if $d(x,y) = max\{d_i(x_i, y_i) : 1 \leqslant i \leqslant n\} < r$, i.e. if and only if $d_i(x_i, y_i) < r$, $1 \leqslant i \leqslant n$. So, $y \in B(x,r)$ if and only if $y_i \in B(x_i, r)$, $1 \leqslant i \leqslant n$, that is, if and only if $y \in \prod_{i=1}^{n} B_i(x_i, r)$.*

Proposition 1.18.

1. *If $O_i \subseteq X$, $1 \leqslant i \leqslant n$ are open subsets in $\mathbb{X}_i$, then $\prod_{i=1}^{n} O_i$ is open in $\mathbb{X}$.*
2. *If $F_i \subseteq X$, $1 \leqslant i \leqslant n$ are closed subsets in $\mathbb{X}_i$, then $\prod_{i=1}^{n} F_i$ is closed in $\mathbb{X}$.*

Proof.

1. *If* $x = (x_1, x_2, \dots, x_n) \in \prod_{i=1}^{n} O_i$, *then there exist positive* $r_1, r_2, \dots, r_n$ *such that* $B(x_i, r_i) \subseteq O_i$, $1 \leqslant i \leqslant n$. *Let* $r = \min\{r_1, r_2, \dots, r_n\}$. *Then,* $B(x, r) = \prod_{i=1}^{n} B_i(x_i, r) \subseteq \prod_{i=1}^{n} O_i$. *Hence,* $\prod_{i=1}^{n} O_i$ *is open.*

2. *Proof left to the reader.*

Example 1.20. *Since* $\mathbb{R}$, (x, y) *and* $(-\infty, z)$ *are open in* $\mathbb{R}$, *then* $(x, y) \times \mathbb{R}$ *and* $(x, y) \times (-\infty, z)$ *are open in* $\mathbb{R}^2$.

1.8 Continuity

1.8.1 Continuous Mappings

Definition 1.21. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be metric spaces and* $A \subseteq \mathbb{X}$. *Then, a function* $f : A \longrightarrow \mathbb{Y}$ *is said to be continuous at* $x_0 \in A$, *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\forall x \in A,\ d_{\mathbb{X}}(x, x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x), f(x_0)) < \varepsilon. \tag{1.10}$$

Remark 1.14. *If* f *is continuous at every point of* A, *then it is said to be continuous on* A.

Example 1.21. *Let* $(\mathbb{X}, d)$ *be a metric space. Every function* $f_a : \mathbb{X} \longrightarrow \mathbb{R}_+$ *defined by* $f_a(x) = d(a, x)$, *such that* $a \in \mathbb{X}$, *is continuous on* $\mathbb{X}$ *because* $|f_a(x) - f_a(y)| = |d(a, x) - d(a, y)| \leqslant d(x, y)$ *(it is enough to take* $\delta = \varepsilon$*).*

Example 1.22. *Let* d_u *and* δ *be the usual metric and the discrete metric on* $\mathbb{R}$, *respectively. Then, the function* $f : (\mathbb{R}, d_u) \longrightarrow (\mathbb{R}, \delta)$ *defined by* $f(x) = x$ *is not continuous on* $\mathbb{R}$ *because if* $x \neq x_0$ *and* $\varepsilon < 1$ *we obtain* $\delta(f(x), f(x_0)) = \delta(x, x_0) = 1$.

Proposition 1.19. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces. A function* $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ *is continuous at a point* $x_0 \in \mathbb{X}$ *if and only if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$B(x_0, \delta) \subseteq f^{-1}(B(f(x_0), \varepsilon)), \tag{1.11}$$

Proof. *The function* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is continuous at* $x_0 \in \mathbb{X}$ *only and only if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that*

$$\forall x \in \mathbb{X},\ d_{\mathbb{X}}(x, x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x), f(x_0)) < \varepsilon,$$

i.e.,

$$x \in B(x_0, \delta) \Longrightarrow f(x) \in B(f(x_0), \varepsilon).$$

or

$$f(B(x_0, \delta)) \subseteq B(f(x_0), \varepsilon).$$

This is equivalent to the condition (1.11).

1.8.2 Uniform Continuity

Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces and let f be a function continuous at each point $x_0 \in \mathbb{X}$. In the definition of continuity, when x_0 and ε are specified, we make a definite choice of δ so that

$$\forall x \in \mathbb{X},\ d_{\mathbb{X}}(x, x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x), f(x_0)) < \varepsilon,$$

This describes δ as dependent upon x_0 and ε, say $\delta = \delta(x_0, \varepsilon)$. If $\delta(x_0, \varepsilon)$ can be chosen in such a way that its values have a lower positive bound when ε is kept fixed and x_0 is allowed to vary over $\mathbb{X}$, and if this happens for each positive ε, then we have the notion of uniform continuity. More precisely, we have the following definition:

Definition 1.22. *Let* $(\mathbb{X}, d_{\mathbb{X}})$, $(\mathbb{Y}, d_{\mathbb{Y}})$ *be tow metric spaces. A function* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is said to be uniformly continuous on* $\mathbb{X}$ *if, for every* ε, *there exists a* δ *(depending on* ε *alone) such that:*

$$\forall x, y \in \mathbb{X},\ d_{\mathbb{X}}(x, y) < \delta(\varepsilon) \Longrightarrow d_{\mathbb{Y}}(f(x), f(y)) < \varepsilon. \tag{1.12}$$

Example 1.23. *The function* $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|)$ *defined by* $f(x) = x$ *is uniformly continuous (it is enough to take* $\delta = \varepsilon$*).*

Using the previous definition we obtain the following result.

Proposition 1.20. *Every uniformly continuous function on* $\mathbb{X}$ *is necessarily continuous on* $\mathbb{X}$. *However, the converse may not be true.*

Example 1.24. *The function* $f:(\mathbb{R},|.|)\longrightarrow(\mathbb{R},|.|)$ *defined by* $f(x)=x^2$ *is continuous but not uniformly continuous. Take* $\varepsilon=1$ *and let* $\delta>0$ *be arbitrary. If we choose* $x=\frac{\delta}{2}+\frac{1}{\delta}$ *and* $y=\frac{1}{\delta}$ *we obtain*

$$|x-y|=\left|\frac{\delta}{2}+\frac{1}{\delta}-\frac{1}{\delta}\right|=\frac{\delta}{2}<\delta,$$

but

$$|f(x)-f(y)|=\left|\left(\frac{\delta}{2}+\frac{1}{\delta})\right)^2-\left(\frac{1}{\delta}\right)^2\right|=1+\frac{\delta^2}{4}>1.$$

1.8.3 Lipschitz and Contraction Mappings and Applications

Definition 1.23. *Let* $(\mathbb{X},d_{\mathbb{X}})$ *and* $(\mathbb{Y},d_{\mathbb{Y}})$ *two metric spaces. A mapping* $f:\mathbb{X}\longrightarrow\mathbb{Y}$ *is said to be k-Lipschitz if there exists a real number* $k>0$ *such that*

$$\forall x,y\in\mathbb{X},\ d_{\mathbb{Y}}(f(x),f(y))\leqslant kd_{\mathbb{X}}(x,y), \tag{1.13}$$

Definition 1.24. *Let* $(\mathbb{X},d_{\mathbb{X}})$ *and* $(\mathbb{Y},d_{\mathbb{Y}})$ *two metric spaces. A mapping* $f:\mathbb{X}\longrightarrow\mathbb{Y}$ *is said to be a contraction (or contraction mapping) if there exists a real number* $0\leqslant k<1$ *such that*

$$\forall x,y\in\mathbb{X},\ d_{\mathbb{Y}}(f(x),f(y))\leqslant kd_{\mathbb{X}}(x,y). \tag{1.14}$$

Proposition 1.21. *Let* $f:\mathbb{R}\supseteq I\longrightarrow\mathbb{R}$ *be a differentiable mapping such that* $|f'(x)|\leqslant k$*, for all* $x\in I$*. Then,* f *is k-Lipschitz. Moreover, if* $|f'(x)|\leqslant k<1$ *for all* $x\in I$*, then* f *is a contraction.*

Proof. *For all* $x,y\in I$*, we have* $|f(x)-f(y)|=|\int_x^y f'(t)dt|\leqslant k|x-y|$*, which implies that* f *is k-Lipschitz. Moreover, if* $|f'(x)|\leqslant k<1$*, then* f *is a contraction.*

Using the previous definition we obtain the following result.

Proposition 1.22. *Every k-Lipschitz or contraction mapping is uniformly continuous.*

1.9 Isometry

Definition 1.25. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces. A bijection* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is called an isometry if*

$$d_{\mathbb{Y}}(f(x), f(y)) = d_{\mathbb{X}}(x, y), \ \forall x, y \in \mathbb{X}. \tag{1.15}$$

In this case, one says that $\mathbb{X}$ *and* $\mathbb{Y}$ *are isometric (or* $\mathbb{X}$ *is isometric to* $\mathbb{Y}$*).*

Remark 1.15. *In other words, an isometry between metric spaces is a bijection which preserves the distance between elements. Clearly,* $\mathbb{Y}$ *is isometric to* $\mathbb{X}$ *if and only if* $\mathbb{X}$ *is isometric to* $\mathbb{Y}$*.*

Example 1.25. *The mapping* $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|)$ *defined by* $f(x) = x \pm b$, $b \in \mathbb{R}$ *is an isometry.*

Example 1.26. *Let* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces such that* $card(\mathbb{X}) = card(\mathbb{Y})$, $d_{\mathbb{X}} = \delta$ *(the discrete distance, see example (1.4)) and* $d_{\mathbb{Y}}(x, y) = \begin{cases} 2 \ si \ x \neq y \\ 0 \ si \ x = y \end{cases}$. *Then,* $(\mathbb{X}, d_{\mathbb{X}})$ *and* $(\mathbb{Y}, d_{\mathbb{Y}})$ *are not isometric because the distance between two different points of the first space is different to the distance between two different points of the second space.*

Remark 1.16. *Every isometry is uniformly continuous (because it is 1-Lipschitz).*

1.10 Normed spaces

In functional analysis, a normed space is a vector space equipped with a norm, which is a function that assigns a non-negative length or size to each vector in the space.

Definition 1.26. *Let* $\mathbb{X}$ *be a vector space over the field* $\mathbb{K}$ *of real or complex numbers. A semi-norm on* $\mathbb{X}$ *is a function* $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ *satisfying the following properties:*

1. *Non-negativity:* $\|x\| \geqslant 0$ *for all* $x \in \mathbb{X}$.
2. *Homogeneity:* $\|\alpha x\| = |\alpha| \|x\|$ *for all* $\alpha \in \mathbb{R}$ *(or* $\mathbb{C}$*) and* $x \in \mathbb{X}$.
3. *Triangle inequality:* $\|x + y\| \leqslant \|x\| + \|y\|$ *for all* $x, y \in \mathbb{X}$.

A semi-normed space $(\mathbb{X}, \|\cdot\|)$ *is a vector space* $\mathbb{X}$ *equipped with a semi-norm.*

Example 1.27.

1. *Both* $\mathbb{R}$ *and* $\mathbb{C}$ *are semi-normed space with* $\|x\| = |x|$.
2. *The function* $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\|(x,y)\| = |x-y|, \tag{1.16}$$

   is a semi-norm.

Proposition 1.23. *Let* $(\mathbb{X}, \|\cdot\|)$ *be a semi-normed space, then*

1. $\|0\| = 0$.
2. $\forall x, y \in \mathbb{X}, \quad \|x-y\| = \|y-x\|$.
3. $\forall x, y \in \mathbb{X}, \quad |\|x\| - \|y\|| \leqslant \|x-y\|$.

Proof.

1. *Let* $\lambda \in \mathbb{K}$ *such that* $\lambda \neq 1$, *then*

$$\|0\| = \|\lambda.0\| = |\lambda| \|0\|,$$

   which implies that

$$\|0\| (1 - |\lambda|) = 0.$$

   Hence, $\|0\| = 0$.

2. *For all* $x, y \in \mathbb{X}$, *we have*

$$\|x-y\| = \|-(y-x)\| = |-1| \|y-x\| = \|y-x\|.$$

3. *Let us write,*

$$y = y - x + x \Longrightarrow \|y\| = \|y-x+x\| \leqslant \|y-x\| + \|x\|,$$

   from which it follows that

$$\|y\| - \|x\| \leqslant \|x-y\| \tag{i}$$

   Similarly, we have

$$x = x - y + y \Longrightarrow \|x\| = \|x-y+y\| \leqslant \|x-y\| + \|y\|,$$

from which it follows that

(ii) $$\|x\| - \|y\| \leqslant \|x-y\|$$

Finally, from inequalities ((i)) and ((ii)) we obtain

$$|\|x\| - \|y\|| \leqslant \|x-y\|.$$

Definition 1.27. *Let $\mathbb{X}$ be a vector space over the field of real or complex numbers. A norm on $\mathbb{X}$ is a semi-norm $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ satisfying, furthermore, the following property:*

$$\|x\| = 0 \Longrightarrow x = 0.$$

A normed space $(\mathbb{X}, \|\cdot\|)$ is a vector space $\mathbb{X}$ equipped with a norm.

Example 1.28.

1. *Both $\mathbb{R}$ and $\mathbb{C}$ are normed space with $\|x\| = |x|$.*

2. *The function defined by (1.16) is not a norm because*

$$\|(1,1)\| = \|1-1\| = 0 \quad \textit{but} \quad (1,1) \neq (0,0).$$

3. *On $\mathbb{R}^n$, for all $x = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$ we have the following norms,*

$$\begin{array}{lrcl} ① & \|x\|_1 & = & \sum\limits_{i=1}^{n} |x_i|, \\ ② & \|x\|_2 & = & \left(\sum\limits_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}, \\ ③ & \|x\|_\infty & = & \max\limits_{1 \leqslant i \leqslant n} |x_i|, \\ ④ & \|x\|_p & = & \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty). \end{array}$$

The corresponding metric space of ③ is denoted by ℓ_∞^n and the corresponding metric space of ④ is denoted by ℓ_p^n.

4. *On the vector space $C([a,b],\mathbb{R})$, for all $f \in C([a,b],\mathbb{R})$ we have the following norms,*

$$\begin{aligned}
\|f\|_1 &= \int_a^b |f(x)|\,dx, \\
\|f(x)\|_2 &= \left(\int_a^b f(x)^2 dx\right)^{\frac{1}{2}}, \\
\|f(x)\|_\infty &= \max_{x\in[a,b]} |f(x)|, \\
\|f(x)\|_p &= \left(\int_a^b |f(x)|^p\,dx\right)^{\frac{1}{p}}, \quad (1 \leqslant p < \infty)
\end{aligned}$$

Normed space are an important instance of metric spaces, as the following proposition asserts.

Proposition 1.24. *Let $(X, \|\cdot\|)$ be a normed space. Then,*

$$d(x,y) = \|x - y\|$$

defines a metric on $\mathbb{X}$. That is, every normed space is automatically a metric space with a canonical metric.

Proof. *Leave to the reader (Immediate).*

Remark 1.17. *Note that metric spaces need not be vector spaces.*

Proposition 1.25. *The norm is a uniformly continuous function.*

Proof. *Using Proposition (1.23)$_3$ we deduce that the norm is 1-Lipschitz.*

Definition 1.28. *Two norms on a $\mathbb{K}$-vector space $\mathbb{X}$ are called equivalent if they define the same open subsets of $\mathbb{X}$.*

Proposition 1.26. *Let $\mathbb{X}$ be a $\mathbb{K}$-vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{X}$ are equivalent if and only if there exist constants $\alpha > 0$ and $\beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1 \quad \textit{for all} \quad x \in \mathbb{X}.$$

Proof. *The proof is left to the readers.*

Example 1.29.

1. *The norms ①, ② and ③ defined in Example(1.28) are equivalent because for all $x \in \mathbb{R}^n$ we have*

$$\|x\|_2 \leqslant \|x\|_1 \leqslant \sqrt{n}\,\|x\|_2\,,$$
$$\|x\|_\infty \leqslant \|x\|_1 \leqslant n\,\|x\|_\infty\,.$$

2. *Let $\mathbb{X}$ the vector space defined by*

$$\mathbb{X} = \left\{ f \in C^1([0,1],) \ \ / \ \ f(0) = 1 \right\},$$

and equipped with the following norms:

$$\|f\|_1 = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} \left|f'(x)\right|, \qquad \|f\|_2 = \sup_{x\in[0,1]} \left|f'(x)\right|.$$

Let us prove that the two previous norms are equivalent. On the one hand, by definition we have

(i) $$\|f\|_2 \leqslant \|f\|_1\,.$$

On the other hand, by application of the finite-increments formula we obtain

$$f(1) - f(x) = f'(c)(1-x), \quad 0 \leqslant x < c < 1.$$

Keeping in mind that $f(1) = 0$ we find

$$f(x) = f'(c)(x-1), \quad 0 \leqslant x < c < 1,$$

from which it follows that

$$\sup_{x\in[0,1]} |f(x)| \leqslant \sup_{x\in[0,1]} \left|f'(x)\right|.$$

Hence,

(ii) $$\|f\|_1 \leqslant 2\,\|f\|_2\,.$$

Finally, from the two previous inequality (i) and (ii) we deduce that $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *are equivalent norms on* $\mathbb{X}$.

We have just seen that norms induce metrics. Next we look at a useful way to induce a norm.

Definition 1.29. *Let* $\mathbb{X}$ *be a vector space over either* $\mathbb{R}$ *or* $\mathbb{C}$. *An inner product (Sesquilinear form) on* $\mathbb{X}$ *is a function* $\langle\cdot,\cdot\rangle : \mathbb{X}\times\mathbb{X}\to\mathbb{K}$, *where* $\mathbb{K}$ *is either* $\mathbb{R}$ *or* $\mathbb{C}$, *such that for all vectors* $x,y,z,w\in\mathbb{X}$ *and all scalars* $\alpha,\beta\in\mathbb{K}$, *the following properties hold:*

1. ***Linearity in the first argument***: $\langle\alpha x+\beta y,z\rangle=\alpha\langle x,z\rangle+\beta\langle y,z\rangle$,
2. ***Conjugate linearity in the second argument:*** $\langle x,\alpha z+\beta w\rangle=\overline{\alpha}\langle x,z\rangle+\overline{\beta}\langle x,w\rangle$,
3. ***Conjugate symmetry***: $\langle x,y\rangle=\overline{\langle y,x\rangle}$ *(for complex vector spaces),*
4. ***Positive-definiteness***: $\langle x,x\rangle\geq 0$ *and* $\langle x,x\rangle=0$ *if and only if* $x=0$.

We present two examples of inner products next; the reader is asked to verify that they satisfy the inner product axioms in definition(1.29).

Example 1.30. *On* $\mathbb{R}^n$, *the function* $\langle\cdot,\cdot\rangle$ *defined by*

$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{i=1}^{n}x_iy_i,$$

is an inner product. When we consider $\mathbb{R}^2$ *or* $\mathbb{R}^3$, *this is often called the dot product.*

Example 1.31. *On the vector space* $C([0,1])$, *the function* $\langle\cdot,\cdot\rangle$ *defined by*

$$\langle f,g\rangle=\int_0^1 f(t)g(t)\,dt,$$

is an inner product.

Our next result is incredibly useful.

Proposition 1.27. ***(Cauchy-Schwarz inequality)*** *Suppose that* $\langle\cdot,\cdot\rangle$ *is an inner product on a real vector space* $\mathbb{X}$. *Then for all* $x,y\in\mathbb{X}$, *we have*

$$\langle x,y\rangle^2\leq\langle x,x\rangle\langle y,y\rangle. \qquad (1.17)$$

Proof. *Let $x, y \in \mathbb{X}$. For all $\lambda \in \mathbb{K}$ we have,*

$$\langle \lambda x + y, \lambda x + y \rangle = \lambda\overline{\lambda}\langle x, x\rangle + \langle y, y\rangle + \lambda\langle x, y\rangle + \overline{\lambda}\langle y, x\rangle \geqslant 0. \tag{1.18}$$

By taking $a = \langle x, x\rangle$, $b = \langle x, y\rangle$ and $c = \langle y, y\rangle$ into (1.18), we obtain

$$a\lambda\overline{\lambda} + b\lambda + \overline{b\lambda} + c \geqslant 0. \tag{1.19}$$

If $a = c = 0$, we set $\lambda = -\overline{b}$ and by substitution into (1.19) we find,

$$-b\overline{b} - b\overline{b} = -2\,|b|^2 \geqslant 0,$$

which implies that $b = 0$. Hence, the inequality (1.17) is verified.
If $a \neq 0$, we set $\lambda = -\dfrac{\overline{b}}{a}$ and by substitution into (1.19) we find,

$$a\left(-\frac{\overline{b}}{a}\right)\left(-\frac{b}{a}\right) - \frac{\overline{b}}{a}b - \frac{\overline{b\overline{b}}}{a} + c \geqslant 0,$$

i.e.,

$$-\frac{|b|^2}{a} + c \geqslant 0,$$

which implies that

$$|b|^2 \leqslant ac,$$

Hence, the inequality (1.17) is verified.

Proposition 1.28. ***(Minkowski inequality)*** *If $\langle \cdot, \cdot \rangle$ is an inner product on the vector space $\mathbb{X}$, then we have*

$$\forall x, y \in \mathbb{X}, \quad \sqrt{\langle x+y, x+y\rangle} \leqslant \sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}. \tag{1.20}$$

Proof. *We know that*

$$\langle x+y, x+y\rangle = \langle x, x\rangle + 2Re\langle x, y\rangle + \langle y, y\rangle,$$

Furthermore, we have

$$Re\langle x, y\rangle \leqslant |\langle x, y\rangle| \leqslant \sqrt{\langle x, x\rangle}.\sqrt{\langle y, y\rangle},$$

which implies that

$$\langle x+y, x+y\rangle \leqslant \left(\sqrt{\langle x, x\rangle} + \sqrt{\langle y, y\rangle}\right)^2.$$

From which the inequality (1.20) follows by taking square roots.

With the *Cauchy-Schwarz* inequality in hand, our final result of the section shows how inner products induce norms (which then induce metrics).

Proposition 1.29. ***(Inner Products Induce Norms)*** *If $\langle\cdot,\cdot\rangle$ is an inner product on the vector space $\mathbb{X}$, then the function $\|\cdot\| : \mathbb{X} \to \mathbb{R}$ defined by*

$$\|x\| = \sqrt{\langle x, x\rangle}, \tag{1.21}$$

is a norm on $\mathbb{X}$.

Proof. *It suffices to use the Minkowski inequality to obtain the triangle inequality. The other properties are left to the reader.*

Definition 1.30. *A pre-Hilbert space (or an inner product space) is a vector space with a norm induced by an inner product.*

Example 1.32.

1. $\mathbb{X} = \mathbb{C}^n$ *is a pre-Hilbert space (or an inner product space) with the following inner product*

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y}_i.$$

2. $\mathbb{X} = C([0,1], \mathbb{C})$ *is a pre-Hilbert space (or an inner product space) with the following inner product*

$$\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}\,dt.$$