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FACULTY OF SCIENCES

# *INTRODUCTION TO METRIC & TOPOLOGICAL SPACES*

## *FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS*


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# CHAPTER 2

# COMPLETE METRIC SPACES

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You are familiar with the notion of a convergent sequence of real numbers. It is defined as follows: the sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to converge to the real number $x$ if given any $\varepsilon > 0$ there exists $n_0$ such that for all $n \geqslant n_0$, $|x_n - x| < \varepsilon$.

It is obvious how this definition can be extended from $\mathbb{R}$ with the Euclidean metric to any metric space.

## 2.1 Convergence in a metric space

### 2.1.1 Convergence and limits

**Definition 2.1.** *A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space $(\mathbb{X}, d)$ is called convergent to $x_0 \in \mathbb{X}$ if,*

$$\forall \varepsilon > 0,\ \exists N_0(\varepsilon) \in \mathbb{N}\ /\ \forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x_n, x_0) < \varepsilon. \tag{2.1}$$

*The point $x$ is called the limit of the sequence $(x_n)_{n\in\mathbb{N}}$, and we write*

$$\lim_{n\longrightarrow+\infty}(x_n) = x_0 \ \text{ or } \ x_n \longrightarrow x_0.$$

*If the sequence does not converge, then it is said to diverge.*

**Remark 2.1.** *The condition (2.1) means that from a certain rank $N_0$ the elements of the sequence $(x_n)$ are in the open ball $B(x_0, \varepsilon)$. Thus, this ball contains an infinite elements of this sequence.*

**Example 2.1.** *In the metric space* $(\mathbb{R}, |.|)$, *the sequence* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}$ *converge to* $0$ *and we write* $\lim_{n\longrightarrow+\infty}\left(\frac{1}{n}\right) = 0.$

**Proof.** *Let* $\varepsilon > 0$ *be given. By the Archimedean property, there is an integer* $N_0 \in \mathbb{N}$ *such that* $N_0 > \frac{1}{\varepsilon}$, *and thus for all* $n \geqslant N_0$, *we have* $d_u\left(\frac{1}{n}, 0\right) = \left|\frac{1}{n}\right| < \varepsilon$.

However, this sequence can be made to diverge by changing the metric on $\mathbb{R}$.

**Example 2.2.** *In the metric space* $(\mathbb{R}, d_{disc})$, *the sequence* $\left(\frac{1}{n}\right)_{n\in\mathbb{N}^*}$ *diverges.*

**Proof.** *Using the definition of the discrete distance (??) we obtain* $\delta\left(\frac{1}{n}, 0\right) = 1$ *because* $\frac{1}{n} \neq 0$. *Hence, if we take* $\varepsilon < 1$, *we obtain* $\delta(\frac{1}{n}, 0) > \varepsilon$.

**Proposition 2.1.** *Let* $(\mathbb{X}, d)$ *be a metric space. The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* $x_0 \in \mathbb{X}$ *if and only if the sequence* $(d(x_n, x_0))$ *converges to* $0$ *in* $(\mathbb{R}, d_u)$.

**Proof.** *Suppose first that* $x_n \longrightarrow x_0$ *in* $(\mathbb{X}, d)$. *Then for any* $\varepsilon > 0$, *there is an* $N_0 \in \mathbb{N}$ *such that* $d(x_n, x_0) = |d(x_n, x_0)| < \varepsilon$ *for all* $n \geqslant N_0$, *but this precisely the same as* $d(x_n, x_0) \longrightarrow 0$ *in* $(\mathbb{R}, d_u)$. *Conversely, if* $d(x_n, x_0) \longrightarrow 0$ *in* $(\mathbb{R}, d_u)$, *then for every* $\varepsilon > 0$ *there is an* $N_0 \in \mathbb{N}$ *such that* $d(x_n, x_0) = |d(x_n, x_0)| < \varepsilon$ *for all* $n \geqslant N_0$, *and this precisely what it means to have* $x_n \longrightarrow x_0$ *in* $(\mathbb{X}, d)$.

**Example 2.3.** *The sequence* $(x_n)_{n\in\mathbb{N}^*}$ *defined by* $x_n = \left(\frac{1}{n}, \frac{1}{n}\right)$ *converges to* $(0, 0)$ *in the three metric spaces* $(\mathbb{R}^2, d_1)$, $(\mathbb{R}^2, d_2)$ *and* $(\mathbb{R}^2, d_\infty)$.

**Proof.** *We have*

$$\begin{aligned} d_1\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0, 0)\right) &= \frac{2}{n}, \\ d_2\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0, 0)\right) &= \frac{\sqrt{2}}{n}, \\ d_\infty\left(\left(\frac{1}{n}, \frac{1}{n}\right), (0, 0)\right) &= \frac{1}{n}. \end{aligned}$$

*The result follows immediately from Proposition(2.1).*

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**Proposition 2.2.** *If a sequence converges then its limit is unique.*

**Proof**. *Suppose that the sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to both* $x$ *and* $y$*. Then, for every* $\varepsilon > 0$ *there exists a (sufficiently large)* $N_0 \in \mathbb{N}$ *such that*

$$\forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x_n, x) < \frac{\varepsilon}{2} \text{ and } d(x_n, y) < \frac{\varepsilon}{2}.$$

*from which we conclude that for every* $\varepsilon > 0$,

$$\forall n \in \mathbb{N},\ n \geqslant N_0 \Longrightarrow d(x,y) < d(x,x_n) + d(x_n,y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

*i.e.,* $d(x,y) = 0$*, namely* $x = y$.

**Proposition 2.3.** *Let* $F$ *be a subset of a metric space* $(\mathbb{X}, d)$ *and* $\mathcal{L}$ *is the set of all* $x \in \mathbb{X}$ *such that* $x$ *is the limit of some sequence of elements of* $F$*. Then,* $F$ *is closed if and only if* $F = \mathcal{L}$.

**Proof**.

$\Longrightarrow$ *) Suppose that* $F$ *is closed.*

• $\mathcal{L} \subseteq F$ ?) *Take* $x \in \mathcal{L}$*, meaning that* $x$ *is a limit of a sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of* $F$*. If* $x \notin F$*, then* $x \in \complement_{\mathbb{X}} F$ *(which is an open), implying that there exists* $r > 0$ *such that* $B(x,r) \subseteq \complement_{\mathbb{X}} F$*. Since* $x_n \longrightarrow x$*, there exists* $N_0 \in \mathbb{N}$ *such that for all,* $n \geqslant N_0$ *we have* $d(x_n, x) < r$*. This means* $x_n \in B(x,r) \subseteq \complement_{\mathbb{X}} F$*, which contradicts the fact that* $x_n \in F$.

• $F \subseteq \mathcal{L}$ ?) *Let* $x \in F$*, then we can consider* $x$ *as a limit of the constant sequence* $x_n = x$ *which implies that* $x \in \mathcal{L}$.

$\Longleftarrow$ *) Suppose that the limit of all convergent sequence of* $F$ *belongs to* $F$*. Let* $x \in Cl(F)$*. Then* $B(x,r) \cap F \neq \emptyset$ *for all* $r > 0$*. Thus, for all* $n \in \mathbb{N}^*$*, there exists* $x_n$ *such that* $x_n \in B(x, \frac{1}{n}) \cap F$*. Then,* $(x_n)$ *is a sequence of elements of* $F$ *that satisfies* $d(x_n, x) < \frac{1}{n}$ *for all* $n \in \mathbb{N}^*$*, which implies,* $x_n \longrightarrow x$*. Therefore,* $x \in F$ *(by hypotheses), which implies that* $F$ *is closed.*

Using the previous proposition we obtain the following result.

**Proposition 2.4.** *Let* $(\mathbb{X}, d)$ *be a metric space and* $F$ *be a subset of* $\mathbb{X}$*, then :*

$$Cl(F) = \left\{ x \in \mathbb{X} : \exists\ (x_n) \subset F \ / \lim_{n \longrightarrow +\infty} x_n = x \right\} \tag{2.2}$$

**Proposition 2.5** (Sequential continuity)**.** *Let* $(\mathbb{X}, d_{\mathbb{X}})$*,* $(\mathbb{Y}, d_{\mathbb{Y}})$ *be two metric spaces and* $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$*. Then,* $f$ *is continuous at* $x_0$ *if and only if for every sequence* $(x_n)_{n\in\mathbb{N}} \subset \mathbb{X}$*, we have*

$$\lim_{n \longrightarrow +\infty} x_n = x_0 \Longrightarrow \lim_{n \longrightarrow +\infty} f(x_n) = f(x_0) \quad (\ f \textit{ is sequentially continuous}).$$

**Proof.**

$\Longrightarrow$) *Suppose that* $f$ *is continuous at* $x_0$ *and let* $(x_n)$ *be a sequence in* $\mathbb{X}$ *that converges to* $x_0$*. Since* $x_n \longrightarrow x_0$*,*

$$\forall \varepsilon > 0,\ \exists n_0(\varepsilon) \in \mathbb{N} \ / \ \forall n \in \mathbb{N},\ n \geqslant n_0 \Longrightarrow d_{\mathbb{X}}(x_n, x_0) < \varepsilon.$$

*Since* $f$ *is continuous at* $x_0$*,*

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon, x_0) \ / \ \forall x \in \mathbb{X},\ d_{\mathbb{X}}(x, x_0) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x), f(x_0)) < \varepsilon.$$

*Then, it is enough to take* $\varepsilon = \delta$ *to obtain,*

$$\forall\ n \geqslant n_0,\ d_{\mathbb{X}}(x_n, x_0) < \delta = \varepsilon \Longrightarrow d_{\mathbb{Y}}(f(x_n), f(x_0)) < \varepsilon,$$

*which shows that* $f$ *is sequentially continuous.*

$\Longleftarrow$) *Suppose that* $f$ *is not continuous at* $x_0$*. Then, there exists* $\varepsilon > 0$ *such that for all* $\delta > 0$*, there exists* $x_\delta$ *that satisfying,*

$$d_{\mathbb{X}}(x_\delta, x_0) < \delta \ \textit{et}\ d_{\mathbb{Y}}(f(x_\delta), f(x_0)) \geqslant \varepsilon.$$

*Thus, for* $\delta = \dfrac{1}{n}$ *there exists a sequence* $(x_n)$ *such that:*

$$d_{\mathbb{X}}(x_n, x_0) < \frac{1}{n} \ \textit{et}\ d_{\mathbb{Y}}(f(x_n), f(x_0)) \geqslant \varepsilon.$$

*This shows that* $(x_n)$ *converges to* $x_0$*, but* $(f(x_n))$ *does not converge to* $f(x_0)$*. Therefore* $f$ *does not sequentially continuous at* $x_0$*.*

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**Remark 2.2.** *In the third chapter (Topological Spaces), we will demonstrate that in the more general context of topological spaces, continuity always implies sequential continuity; however, the converse is not true.*

## 2.2 Cauchy sequences and completeness

The definition of *Cauchy sequences* in general metric spaces is a straightforward generalization of their definition in the real line.

**Definition 2.2.** *Let $(\mathbb{X}, d)$ be a metric space. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of $\mathbb{X}$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geqslant n_0$. In other words,*

$$\forall \varepsilon > 0,\ \exists\ n_0(\varepsilon) \in \mathbb{N}\ /\ \forall (n,m) \in \mathbb{N}^2,\ n,m \geqslant n_0 \Longrightarrow d(x_n, x_m) < \varepsilon. \tag{2.3}$$

**Example 2.4.**

1. *The sequence $(x_n)_{n\geqslant 1}$, where $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, does not satisfy Cauchy's criterion for convergence. Indeed, we have*

$$|x_{2n} - x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \geqslant \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} = \frac{1}{2}.$$

*Thus, it is not the case that $|x_n - x_m| \to 0$ as $n$ and $m$ become large.*

2. *In $(C[0,1], \mathbb{R})$, the sequence $(f_n)_{n\geqslant 1}$ given by*

$$f_n(x) = \frac{nx}{n+x}, \quad x \in [0,1],$$

*is Cauchy in the uniform metric. For $m \geqslant n$, the difference between the functions is given by*

$$f_m(x) - f_n(x) = \frac{mx}{m+x} - \frac{nx}{n+x} = \frac{(m-n)x^2}{(m+x)(n+x)}.$$

*Since this function is continuous on $[0,1]$, it attains its maximum at some point $x_0 \in [0,1]$. Thus, we have*

$$d(f_m, f_n) = \sup_{x\in[0,1]} |f_m(x) - f_n(x)| = \frac{(m-n)x_0^2}{(m+x_0)(n+x_0)} \leq \frac{x_0^2}{n+x_0} \leqslant \frac{1}{n} \to 0,$$

*for large $m$ and $n$.*

3. *If $(x_n)$ is a Cauchy sequence in the discrete metric space $(\mathbb{X}, \delta)$, then $\delta(x_n, x_m) < \varepsilon \Longrightarrow x_n = x_m$ for any $\varepsilon \leqslant 1$. Thus, the sequence $(x_n)$ is convergent.*

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**Proposition 2.6.** *In a metric space* $(\mathbb{X},d)$*, we have:*

1. *Every convergent sequence is a Cauchy sequence.*
2. *Every Cauchy sequence is bounded.*
3. *Every subsequence of a Cauchy sequence is also a Cauchy sequence.*
4. *Every Cauchy sequence that has a convergent subsequence is convergent.*

**Proof.**

1. *If* $x_n \longrightarrow x_0$*, then for every* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geqslant n_0$*, we have* $d(x_n,x_0) < \frac{\varepsilon}{2}$*. Therefore, if* $n \geqslant n_0$ *and* $m \geqslant n_0$*, we obtain*
$$d(x_n,x_m) \leqslant d(x_n,x_0) + d(x_m,x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
2. *If* $(x_n)$ *is a Cauchy sequence, then for* $\varepsilon = 1$*, there exists* $n_0 \in \mathbb{N}$ *such that* $d(x_n,x_{n_0}) < 1$ *for all* $n \geq n_0$*. Let* $r = \max\left(d(x_{n_0},x_1),\dots,d(x_{n_0},x_{n_0-1}),1\right)$*. Then, for all* $n \in \mathbb{N}$*, we have* $d(x_n,x_{n_0}) < r$*, which implies that* $(x_n) \subset B(x_{n_0},r)$*.*
3. *Obvious.*
4. *Suppose* $x_{n_k} \longrightarrow x_0$*. Then, for every* $\varepsilon > 0$*, there exists* $n_1 \in \mathbb{N}$ *such that for all* $n_k \geqslant n_1$*, we have* $d(x_{n_k},x_0) < \frac{\varepsilon}{2}$*. Since* $(x_n)$ *is a Cauchy sequence, there exists* $n_2 \in \mathbb{N}$ *such that for all* $n,m \geqslant n_2$*, we have* $d(x_n,x_m) < \frac{\varepsilon}{2}$*. Let* $n_0 = \max(n_1,n_2)$ *and choose* $k$ *such that* $n_k \geqslant n_0$ *to obtain:*
$$\forall n \in \mathbb{N},\ n \geq n_0 \Longrightarrow d(x_n,x_0) \leq d(x_n,x_{n_k}) + d(x_{n_k},x_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
*Thus,* $x_n \longrightarrow x_0$*.*

**Definition 2.3.** *A metric space* $(\mathbb{X},d)$ *is said to be* *complete* *if every Cauchy sequence in* $(\mathbb{X},d)$ *converges to a limit that is also in* $\mathbb{X}$*.*

**Example 2.5.**

1. *The space* $(\mathbb{R},|.|)$ *is complete because every Cauchy sequence is convergent in* $\mathbb{R}$*.*
2. *The Cauchy sequence* $\left(\frac{1}{n}\right)$ *does not converge in* $(0,1)$*, and therefore* $(0,1)$ *is not complete.*

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3. *In a discrete metric space, every Cauchy sequence is convergent (see Example 2.4(3)), and therefore every discrete metric space is complete.*

**Proposition 2.7.** *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces. If $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ is uniformly continuous and $(x_n)$ is a Cauchy sequence in $(\mathbb{X}, d_{\mathbb{X}})$, then $(f(x_n))$ is a Cauchy sequence in $(\mathbb{Y}, d_{\mathbb{Y}})$.*

**Proof**. *If $(x_n)$ is a Cauchy sequence in $(\mathbb{X}, d_{\mathbb{X}})$, then we have:*

$$\forall \varepsilon > 0,\ \exists\ n_0(\varepsilon) \in \mathbb{N} \textit{ such that } \forall (n,m) \in \mathbb{N}^2,\ n,m \geqslant n_0 \Longrightarrow d_{\mathbb{X}}(x_n, x_m) < \varepsilon.$$

*Since $f$ is uniformly continuous, for $\varepsilon = \delta$, we obtain :*

$$\forall (n,m) \in \mathbb{N}^2,\ n,m \geq n_0 \Longrightarrow d_{\mathbb{Y}}(f(x_n), f(x_m)) < \varepsilon.$$

*Therefore $(f(x_n))$ is a Cauchy sequence in $\mathbb{Y}$.*

**Remark 2.3.** *The previous proposition is false if $f$ is only continuous. For example, consider the function $f : ((-1,1), |.|) \longrightarrow (\mathbb{R}, |.|)$ defined by $f(x) = \dfrac{x}{1-|x|}$. Let $x_n = 1 - \dfrac{1}{n}$. The sequence $(f(x_n))$ is not a Cauchy sequence.*

**Proposition 2.8.** *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces, and let $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ be a uniformly continuous homeomorphism. If $(\mathbb{Y}, d_{\mathbb{Y}})$ is complete, then $(\mathbb{X}, d_{\mathbb{X}})$ is also complete.*

**Proof**. *Let $(x_n)$ be a Cauchy sequence in $\mathbb{X}$. Then, by Proposition (2.7), $(f(x_n))$ is a Cauchy sequence in $\mathbb{Y}$. Since $\mathbb{Y}$ is a complete space, we conclude that $(f(x_n))$ is convergent. This implies the convergence of $(x_n)$ in $\mathbb{X}$ because $f^{-1}$ is continuous.*

**Remark 2.4.** *The converse of the previous proposition is not true.*

**Example 2.6.** *Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces defined as follows:*

- *$\mathbb{X} = [0,1]$ is the closed interval in $\mathbb{R}$, with the standard metric $d_{\mathbb{X}}(x,y) = |x-y|$. This space is complete because every Cauchy sequence in $[0,1]$ converges to a point in $[0,1]$.*
- *$\mathbb{Y} = (0,1)$ is the open interval in $\mathbb{R}$, with the standard metric $d_{\mathbb{Y}}(x,y) = |x-y|$. This space is not complete. For example, the sequence $x_n = \frac{1}{n}$ is a Cauchy sequence in $(0,1)$, but it converges*

*to* $0$*, which is not in* $(0,1)$*.*
*Now define a homeomorphism* $f:[0,1]\to(0,1)$ *by:*

$$f(x)=\frac{x}{2}+\frac{1}{4}.$$

*This function is uniformly continuous and is a homeomorphism because it is continuous, bijective, and its inverse is also continuous. Thus, the converse of the proposition is false: even though* $(\mathbb{X},d_{\mathbb{X}})$ *is complete and* $f$ *is a uniformly continuous homeomorphism,* $(\mathbb{Y},d_{\mathbb{Y}})$ *is not complete.*

Using the previous proposition, we obtain the following result.

**Proposition 2.9.** *If* $(\mathbb{X},d_{\mathbb{X}})$ *and* $(\mathbb{Y},d_{\mathbb{Y}})$ *are two isometric spaces, then* $(\mathbb{X},d_{\mathbb{X}})$ *is complete if and only if* $(\mathbb{Y},d_{\mathbb{Y}})$ *is complete.*

**Proof**. *Evident, because every isometry and its inverse are uniformly continuous homeomorphisms.*

**Proposition 2.10.** *.*

1. *Every complete subset in a metric space* $(\mathbb{X},d_{\mathbb{X}})$ *is closed.*
2. *Every closed subset in a complete metric space* $(\mathbb{X},d_{\mathbb{X}})$ *is complete.*

**Proof**.

1. *Let* $A$ *be a complete subset of* $\mathbb{X}$*, and let* $x\in Cl(A)$*. Then there exists a sequence* $(x_n)$ *of elements in* $A$ *such that* $x_n\longrightarrow x$ *(see Proposition (2.4)). Since* $(x_n)$ *is a Cauchy sequence in* $A$*, and* $A$ *is complete, it follows that* $x\in A$*. This shows that* $A$ *is closed.*
2. *Let* $(\mathbb{X},d_{\mathbb{X}})$ *be a complete metric space, and let* $(x_n)$ *be a Cauchy sequence in a closed subset* $A\subset\mathbb{X}$*. Then* $(x_n)$ *is a Cauchy sequence in* $\mathbb{X}$*, which is complete, so* $x_n\longrightarrow x\in\mathbb{X}$*. Given that* $A$ *is closed, we deduce that* $x\in A$*, which shows that* $A$ *is complete.*

Using the previous proposition, we obtain the following result.

**Proposition 2.11.** *Let $\mathbb{X}$ be a complete metric space, and let $A$ be a subset of $\mathbb{X}$. Then, the metric subspace $(A, d_A)$ is complete if and only if $A$ is closed in $\mathbb{X}$.*

**Example 2.7.**

1. *The intervals $(a,b)$, $(a,+\infty)$, and $(-\infty,b)$ are not complete because they are not closed.*

2. *The intervals $[a,b]$, $[a,+\infty)$, and $(-\infty,b]$ are complete because they are closed in $\mathbb{R}$.*

**Proposition 2.12.** *The product of a finite number of metric spaces is complete if and only if all its factors are complete.*

**Proof**. *Exercise.*

## 2.3 Contractive mapping theorem

**Definition 2.4.** *Let $\mathbb{X}$ be a set and let $f : \mathbb{X} \to \mathbb{X}$. A point $x \in \mathbb{X}$ is called a fixed point of $f$ if $f(x) = x$.*

**Theorem 2.1 (Banach fixed point theorem).** *Let $(\mathbb{X}, d_{\mathbb{X}})$ be a complete metric space. If $f : \mathbb{X} \longrightarrow \mathbb{X}$ is a contraction (see Definition (**??**)), then it has a unique fixed point $x \in \mathbb{X}$.*

**Proof**. *Consider a recursive sequence given by $x_{n+1} = f(x_n)$ with $x_0 \in \mathbb{X}$. For all $n, m \in \mathbb{N}$, if we assume that $n > m$, we obtain:*

$$d(x_n, x_m) \leqslant \sum_{\ell=m}^{n-1} d(x_{\ell+1}, x_\ell) = \sum_{\ell=m}^{n-1} d\left(f^\ell(x_1), f^\ell(x_0)\right). \tag{1}$$

*On the other hand, using the contraction property, we obtain:*

$$d\left(f^\ell(x_1), f^\ell(x_0)\right) \leqslant k d\left(f^{\ell-1}(x_1), f^{\ell-1}(x_0)\right) \leqslant k^\ell d(x_1, x_0). \tag{2}$$

*Now, keeping in mind that $0 \leqslant k < 1$ and using **(1)** and **(2)**, we obtain:*

$$d(x_n, x_m) \leqslant \sum_{\ell=m}^{n-1} k^{\ell} d(x_1, x_0) = k^m \frac{1-k^{n-m-1}}{1-k} d(x_1, x_0) \leqslant \frac{k^m}{1-k} d(x_1, x_0). \tag{3}$$

*Since $\lim\limits_{m\to\infty} \dfrac{k^m}{1-k} = 0$, we conclude from inequality **(3)** that $(x_n)$ is a Cauchy sequence and therefore convergent to $x \in \mathbb{X}$ because $(\mathbb{X}, d_{\mathbb{X}})$ is complete. Since $f$ is continuous, we obtain:*

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x.$$

*Thus, $x$ is a fixed point of $f$.*
*For uniqueness, suppose that $x_1$ and $x_2$ are two fixed points of $f$. Then we have:*

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leqslant k d(x_1, x_2) \Longrightarrow (1-k) d(x_1, x_2) \leqslant 0.$$

*Since $0 \leqslant k < 1$, we conclude from the last inequality that $x_1 = x_2$.*

**Remark 2.5.**

1. *A function having one or multiple fixed points does not imply that it is a contracting function.*

2. *The assumption that " $f$ is contracting " cannot generally be replaced by the weaker assumption $d(f(x), f(y)) \leqslant d(x, y)$ for all $x \neq y$, as demonstrated by the following example:*

   $$f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, |.|) \text{ such that } f(x) = \sqrt{1+x^2}.$$

3. *The assumption that "$\mathbb{X}$ is complete" is fundamental. For example, if $\mathbb{X} = \left(0, \frac{1}{4}\right)$ (which is not complete) and $f : \mathbb{X} \longrightarrow \mathbb{X}$ is defined by $f(x) = x^2$, then $f$ is a contraction on $\mathbb{X}$ that has no fixed point in $\mathbb{X}$.*

This theorem can be easily generalized in the following way.

**Theorem 2.2.** *Let $(\mathbb{X}, d_{\mathbb{X}})$ be a complete metric space and let $f : \mathbb{X} \longrightarrow \mathbb{X}$. If there exists $n \in \mathbb{N}^*$ such that $f^{(n)}$ is a contraction, then $f$ has a unique fixed point.*

**Proof.** *Since $(\mathbb{X}, d_{\mathbb{X}})$ is complete and $f^{(n)}$ is a contraction, $f^{(n)}$ has a unique fixed point $x_0 \in \mathbb{X}$. Since $f^{(n)}(f(x_0)) = f\left(f^{(n)}(x_0)\right) = f(x_0)$, it follows, by the uniqueness of the fixed point of $f^{(n)}$, that $f(x_0) = x_0$, and thus $x_0$ is the unique fixed point of $f$.*

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**Proposition 2.13.** *If* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *with* $|f'(x)| \leqslant k < 1$, *then* $f$ *has a unique fixed point.*

**Proof**. *Using Proposition (**??**), we conclude that* $f$ *is a contraction on* $\mathbb{R}$, *and since* $\mathbb{R}$ *is complete, we conclude by Theorem (2.1) that* $f$ *has a unique fixed point.*

**Example 2.8.** *Let* $f : [0,1] \longrightarrow [0,1]$ *be defined by* $f(x) = \frac{e^x}{5}$. *We have* $|f'(x)| = \frac{e^x}{5} \leqslant \frac{e}{5} < 1$, *and* $[0,1]$ *is complete because it is a closed subset of* $\mathbb{R}$, *which is complete. Therefore, using Theorem (2.1), we conclude that* $f$ *has a unique fixed point.*

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