وزارة التعليـم العالـي والبحـث العلمـي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

INTRODUCTION TO METRIC & TOPOLOGICAL SPACES

FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS

Prepared by:

Dr. CHOUGUI Nadhir

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CHAPTER 3

TOPOLOGICAL SPACES

3.1 Topology, Open sets and Closed sets

Let $\mathbb{X}$ be a non-empty set and $\mathcal{P}(\mathbb{X})$ be the power set of $\mathbb{X}$.

Definition 3.1. *A topology on $\mathbb{X}$ is a collection of sets $\mathcal{T} \subseteq \mathcal{P}(\mathbb{X})$ that satisfies :*

A_1) *$\emptyset$ and $\mathbb{X}$ are elements of $\mathcal{T}$,*

A_2) *any union (finite or infinite) of elements of $\mathcal{T}$ is an element of $\mathcal{T}$, that is, for any collection $\{O_i \in \mathcal{T} : \ i \in I\}$ we have $\bigcup\limits_{i \in I} O_i \in \mathcal{T}$,*

A_3) *any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$, that is, for any collection $\{O_i \in \mathcal{T} : \ 1 \leqslant i \leqslant n\}$ we have $\bigcap\limits_{i=1}^{n} O_i \in \mathcal{T}$.*

The pair $(\mathbb{X}, \mathcal{T})$ is called a topological space, and the elements of $\mathcal{T}$ are called open sets of the topology.

Example 3.1. *Let $\mathbb{X} = \{1, 2\}$. The topologies defined on $\mathbb{X}$ are:*

$$\begin{aligned}
\mathcal{T}_1 &= \{\emptyset, \mathbb{X}\}. \\
\mathcal{T}_2 &= \{\emptyset, \mathbb{X}, \{1\}\}. \\
\mathcal{T}_3 &= \{\emptyset, \mathbb{X}, \{2\}\}. \\
\mathcal{T}_4 &= \{\emptyset, \mathbb{X}, \{1\}, \{2\}\}.
\end{aligned}$$

Example 3.2. *Let $\mathbb{X} = \{x, y, z, t, s, w\}$ and $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z, t\}, \{x, z, t\}, \{y, z, t, s, w\}\}$. Then $\mathcal{T}$ is a topology on $\mathbb{X}$ as it satisfies conditions $(A_1), (A_2)$ and (A_3) of Definition(3.1).*

Example 3.3. *Let $\mathbb{X} = \{x,y,z,t,s\}$ and $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z,t\}, \{x,z,s\}, \{y,z,t\}\}$. Then $\mathcal{T}$ is not a topology on $\mathbb{X}$ as the union $\{z,t\} \bigcup \{x,z,s\} = \{x,z,t,s\}$ of two members of $\mathcal{T}$ does not belong to $\mathcal{T}$; that is, $\mathcal{T}$ does not satisfy condition (A_2) of Definition(3.1).*

Example 3.4. *Let $\mathbb{N}$ the set of all natural numbers and let $\mathcal{T}$ the collection consisting of $\mathbb{N}$, $\emptyset$ and all finite subsets of $\mathbb{N}$. Then $\mathcal{T}$ is not a topology on $\mathbb{N}$, since the infinite union $\{3\} \cup \{4\} \cup \{5\} \cup \cdots \cup \{n\} \cup \cdots = \{3,4,5,\ldots,\{n\},\ldots\}$ of members of $\mathcal{T}$ does not belong to $\mathcal{T}$; that is, $\mathcal{T}$ does not have property (A_2) of Definition(3.1).*

Definition 3.2. *Let $\mathbb{X}$ be any non-empty set and $\mathcal{T}$ the collection of all sets of $\mathbb{X}$ (the power set of $\mathbb{X}$). Then $\mathcal{T}$ is called the discrete topology on the set $\mathbb{X}$ and is denoted by $\mathcal{T}_{Disc}$. The topological space $(\mathbb{X}, \mathcal{T}_{Disc})$ is called a discrete space.*

Definition 3.3. *Let $\mathbb{X}$ be any non-empty set and $\mathcal{T} = \{\mathbb{X}, \emptyset\}$. Then $\mathcal{T}$ is called the indiscrete topology or trivial topology and is denoted by $\mathcal{T}_{Ind}$. The topological space $(\mathbb{X}, \mathcal{T}_{Ind})$ is called an indiscrete space.*

Remark 3.1. *Every set indeed admits at least two topologies.*

Definition 3.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A subset F of $\mathbb{X}$ is said to be a closed set in $(\mathbb{X}, \mathcal{T})$ if its complement, namely $\complement_{\mathbb{X}} F$ or $\mathbb{X} \setminus F$, is open in $(\mathbb{X}, \mathcal{T})$. We denote by $\mathcal{F}$ the set of all closed subsets in $(\mathbb{X}, \mathcal{T})$.*

Example 3.5. *In Example (3.1), if we consider the topology $\mathcal{T}_2$, then the set $\{2\}$ is closed.*

Example 3.6. *In Example(3.2), the closed sets are*

$$\mathcal{F} = \{\emptyset, \mathbb{X}, \{y,z,t,s,w\}, \{x,y,s,w\}, \{y,s,w\}, \{x\}\}.$$

.

Example 3.7. *Let $\mathbb{X} = (\mathbb{R}, |\cdot|)$. Then $\mathbb{N}$ and $\mathbb{Z}$ are closed.*

Proposition 3.1. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, the collection $\mathcal{F}$ of closed sets in $\mathbb{X}$ satisfies the following properties:*

P_1) *$\mathbb{X}$ and $\emptyset$ are closed sets,*

P_2) *any finite union of closed sets is closed,*

P_3) *any arbitrary intersection of closed sets is closed.*

Proof. *These properties of closed sets directly follow from the properties verified by open sets in a topology. Indeed:*

- *We have seen that $\mathbb{X}$ and $\emptyset$ are open, and since $\complement_{\mathbb{X}}\emptyset = \mathbb{X}$ and $\complement_{\mathbb{X}}\mathbb{X} = \emptyset$, we conclude that $\mathbb{X}$ and $\emptyset$ are closed. Thus, (P_1) is verified.*
- *Let $\{F_i : i = 1, 2, \ldots, n\}$ be a finite family of closed sets in $\mathbb{X}$. Then, for all $i = 1, 2, \ldots, n$, their complements $\complement_{\mathbb{X}}F_i$ are open sets. But $\complement_{\mathbb{X}}\left(\bigcup\limits_{i=1}^{n} F_i\right) = \bigcap\limits_{i=1}^{n} \complement_{\mathbb{X}}F_i$ is an open set (because it is a finite intersection of open sets). Hence $\bigcup\limits_{i=1}^{n} F_i$ is a closed set. Thus, (P_2) is verified.*
- *Let $\{F_i : i \in I\}$ be any family of closed sets of $\mathbb{X}$. Then, for all $i \in I$, their complements $\complement_{\mathbb{X}}F_i$ are open sets. But $\complement_{\mathbb{X}}\left(\bigcap\limits_{i\in I} F_i\right) = \bigcup\limits_{i\in I} \complement_{\mathbb{X}}F_i$ is an open set (because it is an union of any open sets). Hence, $\bigcap\limits_{i\in I} F_i$ is a closed set. Thus, (P_3) is verified.*

Remark 3.2. *A topology can be defined either by the collection of its open sets or by the collection of its closed sets.*

Remark 3.3. *A subset of a topological space can be both open and closed. Moreover, a subset of a topological space can be neither open nor closed.*

Example 3.8. *If we consider Example(3.2), we see that*

1. *the set $\{x\}$ is both open and closed;*
2. *the set $\{y, z\}$ is neither open nor closed.*

Definition 3.5. *A subset $\mathcal{A}$ of a topological space $(\mathbb{X}, \mathcal{T})$ is said to be clopen if it is both open and closed set in $(\mathbb{X}, \mathcal{T})$.*

Example 3.9.

1. *In a discrete space all subsets in* $(\mathbb{X}, \mathcal{T}_{Dis})$ *are clopen.*
2. *In a indiscrete space the only clopen subsets in* $(\mathbb{X}, \mathcal{T}_{Ind})$ *are* $\mathbb{X}$ *and* $\emptyset$.
3. *In every topological space* $(\mathbb{X}, \mathcal{T})$ *both* $\mathbb{X}$ *and* $\emptyset$ *are clopen.*

Definition 3.6. *Let* $\mathbb{X}$ *be a non-empty set, and*

$$\mathcal{T}_{Cof} = \{O \subseteq \mathbb{X} : \complement_{\mathbb{X}} O \text{ is finite}\} \cup \{\emptyset\}.$$

Then, $(\mathbb{X}, \mathcal{T}_{Cof})$ *is a topology, and is called the cofinite topology on* $\mathbb{X}$.

Once again is necessary to check that $\mathcal{T}_{Cof}$ in the previous definition is indeed a topology; that is, that it satisfies each of the conditions of Definition(3.1).

3.2 Neighborhoods

Definition 3.7. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space. A subset* N_x *of* $\mathbb{X}$ *is called a neighborhood of* x *in* $\mathbb{X}$ *if there exists an open set* O_x *of* $\mathbb{X}$ *such that* $x \in O_x \subseteq N_x$. *The collection of neighborhoods of* x *is denoted by* $\mathcal{N}(x)$ *and is called the neighborhood system at* x.

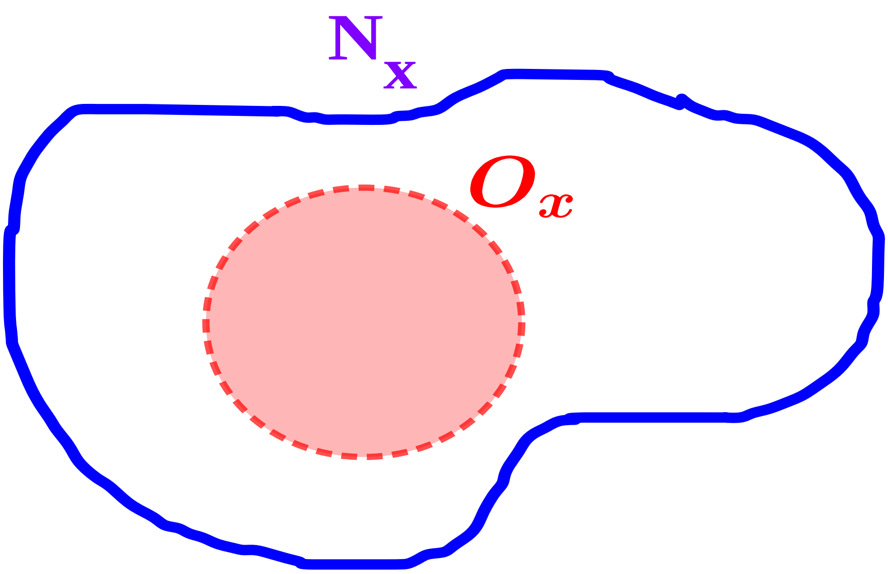

Figure 3.1: Neighborhood $\mathbf{N}_x$

The previous definition can be written in the following form:

$$(N_x \text{ is a neighborhood of } x) \Leftrightarrow (\exists\, O_x \in \mathcal{T} \ / \ x \in O_x \subseteq N_x). \tag{3.1}$$

Definition 3.8. *Let $(\mathbb{X},\mathcal{T})$ be a topological space. We say that a subset N of $\mathbb{X}$ is a neighborhood of a non-empty subset A of $\mathbb{X}$ if there exists an open set O in $\mathcal{T}$ such that $A \subseteq O \subseteq N$. In other words:*

$$(N \text{ is a neighborhood of } A) \Leftrightarrow (\exists O \in \mathcal{T} \text{ such that } A \subseteq O \subseteq N). \tag{3.2}$$

Example 3.10.

1. *Let $(\mathbb{X},\mathcal{T}_{Ind})$. Then, for all $x \in \mathbb{X}$ we have $\mathcal{N}(x) = \{\mathbb{X}\}$.*

2. *Let $(\mathbb{X},\mathcal{T}_{Disc})$ and $x \in \mathbb{X}$. Then, every subset of $\mathbb{X}$ that contains x is an element of $\mathcal{N}(x)$.*

3. *Let $(\mathbb{X},\mathcal{T}) = (\mathbb{R},|.|)$ and $x \in \mathbb{R}$. Then, every subset of $\mathbb{R}$ that contains an interval centered at x is a neighborhood of x.*

4. *Let $\mathbb{X} = \{1,2,3,4\}$ and $\mathcal{T} = \{\emptyset,\mathbb{X},\{1\},\{4\},\{1,4\}\}$. Then we have:*
   - $\mathcal{N}(1) = \{\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\mathbb{X}\}$;
   - $\mathcal{N}(2) = \{\mathbb{X}\}$;
   - $\mathcal{N}(\{1,4\}) = \{\{1,4\},\{1,2,4\},\{1,3,4\},\mathbb{X}\}$.

Remark 3.4. *It follows from the previous definition that if $B \subset A$, then every neighborhood of A is a neighborhood of B.*

Proposition 3.2. *Let $(\mathbb{X},\mathcal{T})$ be a topological space and A a subset of $\mathbb{X}$. Then, we have*

$$(N \text{ is a neighborhood of } A) \Longleftrightarrow (\forall x \in A : N \in \mathcal{N}(x)). \tag{3.3}$$

Proof.

$\Longrightarrow$) *Obvious.*

$\Longleftarrow$) *Suppose that N is a neighborhood of every point in A. Then, we have*

$$\forall x \in A, \exists O_x \in \mathcal{T} \ / \ x \in O_x \subseteq N, \tag{3.4}$$

from which we conclude that $A \subseteq \bigcup\limits_{x \in A} O_x \subseteq N$, and since $\bigcup\limits_{x \in A} O_x \in \mathcal{T}$, it follows that N is a neighborhood of A.

Proposition 3.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A non-empty set A is an open set in $\mathbb{X}$ if and only if A is a neighborhood of each of its points.*

Proof.

$\Longrightarrow$) *Suppose A is an open set in $\mathbb{X}$. Then, using Definition (3.7), we conclude that A is a neighborhood of each of its points.*

$\Longleftarrow$) *Suppose A is a neighborhood of each of its points. Then, for every $x \in A$, there exists $O_x \in \mathcal{T}$ such that $x \in O_x \subseteq A$, hence $A = \bigcup\limits_{x \in A} O_x$. Therefore, A is open as a union of open sets.*

Proposition 3.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. The neighborhoods of a point satisfy the following properties:*

1. *For every $N \in \mathcal{N}(x)$, we have $x \in N$.*
2. *For every $N \in \mathcal{N}(x)$ and every $\mathbf{U} \subset \mathbb{X}$, if $N \subset \mathbf{U}$ then $\mathbf{U} \in \mathcal{N}(x)$.*
3. *Any finite intersection of neighborhoods of x is a neighborhood of x.*
4. *For every $N \in \mathcal{N}(x)$, there exists $W \in \mathcal{N}(x)$ such that for every $a \in W$, we have $N \in \mathcal{N}(a)$.*

Proof.

- *The two properties 1 and 2 are evident.*
- *For the third property, if* $\{N_i : i = 1, \ldots, n\}$ *is a family of neighborhoods of* $x \in \mathbb{X}$, *then for all* $i = 1, \ldots, n$, *there exists* $O_i \in \mathcal{T}$ *such that* $x \in O_i \subseteq N_i$, *from which we conclude that* $x \in \bigcap_{i=1}^{n} O_i \subseteq \bigcap_{i=1}^{n} N_i$. *We deduce that* $\bigcap_{i=1}^{n} N_i \in \mathcal{N}(x)$ *because* $\bigcap_{i=1}^{n} O_i \in \mathcal{T}$.
- *For the fourth property, if* $N \in \mathcal{N}(x)$, *then there exists* $O \in \mathcal{T}$ *such that* $x \in O \subseteq N$. *This implies that* N *is a neighborhood of every point* $a \in O$. *Then, it suffices to take* $W = O$ *to verify that property (4) is holds.*

3.3 Comparison of topologies

Definition 3.9. *Let* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *be two topologies on a set* $\mathbb{X}$. *We say that* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *(or that* $\mathcal{T}_2$ *is coarser than* $\mathcal{T}_1$*) if* $\mathcal{T}_2 \subseteq \mathcal{T}_1$. *In other words,* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *if one of the following three statements holds:*

1. *Every open set in* $(\mathbb{X}, \mathcal{T}_2)$ *is also an open set in* $(\mathbb{X}, \mathcal{T}_1)$.
2. *Every closed set in* $(\mathbb{X}, \mathcal{T}_2)$ *is also a closed set in* $(\mathbb{X}, \mathcal{T}_1)$.
3. *If* $x \in \mathbb{X}$, *then every neighborhood of* x *in* $(\mathbb{X}, \mathcal{T}_2)$ *is also a neighborhood of* x *in* $(\mathbb{X}, \mathcal{T}_1)$.

Remark 3.5. *If* $\mathcal{T}_1$ *is finer than* $\mathcal{T}_2$ *and* $\mathcal{T}_2$ *is finer than* $\mathcal{T}_1$, *we say that* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are equivalent.*

Example 3.11. *For any topological space* $(\mathbb{X}, \mathcal{T})$, *the indiscrete topology on* $\mathbb{X}$ *is coarser than* $\mathcal{T}$ *which in turn is coarser than the discrete topology on* $\mathbb{X}$.

Example 3.12. *The Sierpinski space* $\mathbb{S}$ *consists of two points* $\{0,1\}$ *with the topology* $\{\emptyset, \{1\}, \{0,1\}\}$. *The topology of Sierpinski space is finer than the indiscrete topology* $\mathcal{T}_{Ind} = \{\emptyset, \{0,1\}\}$ *on* $\{0,1\}$ *but coarser than the discrete topology* $\mathcal{T}_{Disc} = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$ *on* $\{0,1\}$.

Example 3.13. *If* $\mathbb{X} = \{x, y, z\}$, *then* $\mathcal{T}_1 = \{\emptyset, \{x\}, \mathbb{X}\}$, $\mathcal{T}_2 = \{\emptyset, \{x, y\}, \mathbb{X}\}$, *and* $\mathcal{T}_3 = \{\emptyset, \{x\}, \{x, y\}, \mathbb{X}\}$ *are three distinct topologies on* $\mathbb{X}$. *The topologies* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are coarser than* $\mathcal{T}_3$; *however,* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *are not comparable.*

Proposition 3.5. *Let $\{\mathcal{T}_i : i \in I\}$ be a collection of topologies on $\mathbb{X}$. Then, $\bigcap_{i\in I} \mathcal{T}_i$ is a topology on $\mathbb{X}$ that is the coarsest of each of the topologies $\mathcal{T}_i$.*

Proof. *Obvious.*

Proposition 3.6. *Let β be a family of subsets of $\mathbb{X}$. There exists a smallest topology that contains β. This topology is called the topology generated by β.*

Proof. *The set of topologies that contain β is not empty because it contains the discrete topology. Therefore, it is enough to take the intersection of these topologies.*

3.4 Base and Neighborhood base

Definition 3.10. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. A* ***basis*** *for the topology $\mathcal{T}$ is a family $\mathfrak{B} \subseteq \mathcal{T}$ such that every set in $\mathcal{T}$ is a union of sets from $\mathfrak{B}$.*

Example 3.14.

1. *Let the topological space $(\mathbb{R}, |.|)$ and $x \in \mathbb{R}$. The collection:*

$$\mathfrak{B} = \{]x, y[\ : x, y \in \mathbb{R}\},$$

is a basis for the usual topology.

2. *In the topological space $(\mathbb{X}, \mathcal{T}_{Disc})$, the collection:*

$$\mathfrak{B} = \{\{x\}\ : x \in \mathbb{X}\},$$

is a basis for the discrete topology.

3. *Let $\mathbb{X} = \{x, y, z\}$ and $\mathcal{T} = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \mathbb{X}\}$. The collection:*

$$\mathfrak{B} = \{\{x\}, \{y\}, \mathbb{X}\},$$

is a basis for this topology.

4. *If $(\mathbb{X}, \mathcal{T})$ is a topological space, then $\mathcal{T}$ is a basis for itself.*

5. *In the topological space $(\mathbb{X}, \mathcal{T}_{Ind})$, the collection:*

$$\mathfrak{B} = \{\mathbb{X}\},$$

is a basis for the indiscrete topology.

Remark 3.6. *If $\mathfrak{B}$ is a basis for a topological space $(\mathbb{X}, \mathcal{T})$ and $\mathfrak{B}'$ is a family that contains $\mathfrak{B}$, then by using the previous definition, we conclude that $\mathfrak{B}'$ is another basis for $\mathcal{T}$. Therefore, a topological space can have multiple bases.*

Proposition 3.7. *Any basis $\mathfrak{B}$ of a topology $\mathcal{T}$ on $\mathbb{X}$ has the following two properties:*

1. *For every $x \in \mathbb{X}$, there exists $B \in \mathfrak{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathfrak{B}$ and $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proof. *Suppose that $\mathfrak{B}$ is a basis of the topology $\mathcal{T}$.*

1. *Since $\mathbb{X}$ is an open set, we have $\mathbb{X} = \bigcup\limits_{B \in \mathfrak{B}} B$ (see definition (3.10)), from which it follows that for every $x \in \mathbb{X}$, there exists $B \in \mathfrak{B}$ such that $x \in B$.*

2. *If $B_1, B_2 \in \mathfrak{B}$, then $B_1, B_2 \in \mathcal{T}$ (since $\mathfrak{B} \subseteq \mathcal{T}$), which implies that $B_1 \cap B_2 \in \mathcal{T}$. Thus, $B_1 \cap B_2 = \bigcup\limits_{B \in \mathfrak{B}} B$ (see definition (3.10)). Therefore, for every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathfrak{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.*

Proposition 3.8. *If $\mathfrak{B}$ is a family of subsets of a set $\mathbb{X}$ that satisfies the two properties of Proposition (3.7), then $\mathcal{T} = \{\bigcup B : B \in \mathfrak{B}\}$ is a topology on $\mathbb{X}$.*

Proof. *We leave it as an exercise.*

Now, using the two previous propositions, we obtain the following result:

Proposition 3.9. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, a family of subsets $\mathfrak{B}$ of $\mathbb{X}$ is a basis for $\mathcal{T}$ if and only if $\mathfrak{B}$ satisfies the two properties of Proposition (3.7).*

Proposition 3.10. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $\mathfrak{B}$ a subset of $\mathcal{T}$. Then, $\mathfrak{B}$ is a basis for $\mathcal{T}$ if and only if for every $O \in \mathcal{T}$ and for every $x \in O$, there exists $U_x \in \mathfrak{B}$ such that: $x \in U_x \subseteq O$.*

Proof.

$\Longleftarrow$) *It is clear that $O = \bigcup_{x \in O} U_x$, hence $\mathfrak{B}$ is a basis for $\mathcal{T}$.*

$\Longrightarrow$) *If $\mathfrak{B}$ is a basis for $\mathcal{T}$, then every subset O of $\mathcal{T}$ is a union of elements of $\mathfrak{B}$, which means that for each element $x \in O$, there exists $U_x \in \mathfrak{B}$ such that $x \in U_x \subset O$.*

Proposition 3.11. *Let $\mathfrak{B}_1$ be a basis of a topology $\mathcal{T}$ and $\mathfrak{B}_2$ a family of subsets of $\mathcal{T}$. If every element of $\mathfrak{B}_1$ is a union of elements of $\mathfrak{B}_2$, then $\mathfrak{B}_2$ is a basis for $\mathcal{T}$.*

Proof. *Let $O \in \mathcal{T}$. Then, there exists $\{O_i : i \in I \text{ and } O_i \in \mathfrak{B}_1\}$ such that $O = \bigcup_{i \in I} O_i$ (because $\mathfrak{B}_1$ is a base for $\mathcal{T}$) and since every element of $\mathfrak{B}_1$ is a union of elements of $\mathfrak{B}_2$, there exists $\{U_{i,j} : j \in J \text{ and } U_{i,j} \in \mathfrak{B}_2\}$ such that $O_i = \bigcup_{j \in J} U_{i,j}$ for all $i \in I$. Thus, we obtain $O = \bigcup_{(i,j) \in I \times J} U_{i,j}$. Therefore, $\mathfrak{B}_2$ is a base for $\mathcal{T}$.*

Definition 3.11. *A collection $\mathcal{S}(x) \subseteq \mathcal{N}(x)$ is called a neighborhood base at x if for every neighborhood N_x, there is a neighborhood $W_x \in S(x)$ such that $W_x \subseteq N_x$. We refer to the sets in $\mathcal{S}(x)$ as basic neighborhoods of x.*

Example 3.15.

1. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, we have:*

$$\mathcal{S}(x) = \{O \in \mathcal{T} : x \in O\}$$

   is a neighbourhoods base of x.

2. *In the topological space $(\mathbb{X}, \mathcal{T}_{Disc})$, we have:*

$$\mathcal{S}(x) = \{\{x\}\}$$

   is a neighborhoods base of x.

3. *Let the topological space* $(\mathbb{R}, |.|)$ *and* $x \in \mathbb{R}$. *Then, we have:*

$$\mathcal{S}(x) = \{(x-\varepsilon, x+\varepsilon) \ : \varepsilon > 0\}$$

is a neighborhoods base of x. *For example:*

$$\mathcal{S}(x) = \left\{\left(x-\frac{1}{n}, x+\frac{1}{n}\right) \ : n \in \mathbb{N}^*\right\}$$

is a countable neighborhoods base of x.

3.5 Interior points, Adherent points, Accumulation points, Isolated points, Boundary points, Exterior points and Dense sets.

3.5.1 Interior points

Definition 3.12. *Let* A *be a subset of a topological space* $(\mathbb{X}, \mathcal{T})$. *We say that* x *is an interior point of* A *if* A *is a neighborhood of* x, *in other words,*

$$x \text{ is an interior point of } A \Longleftrightarrow A \in \mathcal{N}(x). \tag{3.5}$$

The set of all interior points of A *is called the interior or the interior set of* A *and is denoted by* $Int(A)$.

Example 3.16.

1. *Consider the topological space* $(\mathbb{X}, \mathcal{T}_{Ind})$ *and let* $A \subseteq \mathbb{X}$. *Then, we have the following two cases:*
   - $\mathbb{X} = A \Longrightarrow Int(A) = \mathbb{X}$.
   - $\mathbb{X} \neq A \Longrightarrow Int(A) = \emptyset$.
2. *Consider the topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$ *and let* $A \subseteq \mathbb{X}$. *Then,* $Int(A) = A$.
3. *For the topological space* $(\mathbb{R}, |.|)$, *we have:*
   - $\forall x \in \mathbb{R}, \ Int\{x\} = \emptyset$.
   - $\forall x, y \in \mathbb{R}, \ Int([x,y]) = Int([x,y)) = Int((x,y]) = Int((x,y)) = (x,y)$.
   - $Int(\mathbb{N}) = Int(\mathbb{Z}) = Int(\mathbb{Q}) = Int\left(\complement_{\mathbb{R}}\mathbb{Q}\right) = \emptyset$.

4. *If* $\mathbb{X} = \{x, y, z, t\}$ *and* $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{y\}, \{x, y\}\}$, *then we have:*

- $Int\{z\} = Int\{t\} = \emptyset$.
- $Int\{x, z, t\} = \{x\}$.

Proposition 3.12. *$Int(A)$ is the largest open set contained in A.*

Proof. *We will show that $Int(A)$ is the union of all open subsets of A.*
For every $x \in Int(A)$, we have $A \in \mathcal{N}(x)$ (see definition (3.12)). Using definition (3.1), we conclude that: for every $x \in Int(A)$, there exists $O_x \in \mathcal{T}$ such that $x \in O_x \subset A$, which leads to:

$$Int(A) \subset \bigcup_{x \in Int(A)} O_x \subset \bigcup_{x \in A} O_x. \tag{i}$$

Conversely, if $x \in \bigcup_{x \in A} O_x$ then $x \in O_x \subset A$, which implies $A \in \mathcal{N}(x)$, so $x \in Int(A)$. This means that:

$$\bigcup_{x \in A} O_x \subset Int(A). \tag{ii}$$

From (i) and (ii) we conclude that:

$$Int(A) = \bigcup_{x \in A} O_x.$$

Finally, $Int(A)$ is the largest open set contained in A because it is the union of all open subsets of A.

Remark 3.7. *The previous proposition allows us to write the following result:*

$$A \text{ is open in } \mathbb{X} \Longleftrightarrow A = Int(A). \tag{3.6}$$

Proposition 3.13. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A, B two subsets of $\mathbb{X}$. Then, we have:*

1. *If $A \subset B$ and A is open, then $A \subset Int(B)$.*
2. *If $A \subset B$, then $Int(A) \subset Int(B)$.*
3. $Int(A) = Int(Int(A))$.
4. $Int(A \cap B) = Int(A) \cap Int(B)$.

5. $Int(A) \cup Int(B) \subset Int(A \cup B)$.

6. $A \in \mathcal{N}(B) \Longleftrightarrow B \subset Int(A)$.

Proof. *(Exercise).*

Remark 3.8. *We have* $Int\left(\bigcap_{i \in I} A_i\right) \subset \bigcap_{i \in I} Int(A_i)$ *if* I *is infinite.*

3.5.2 Adherent points

Definition 3.13. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$*, and* $x \in \mathbb{X}$*. We say that* x *is an adherent point to* A *if and only if every* $N \in \mathcal{N}(x)$ *contains at least one point of* A*. In other words:*

$$x \text{ is an adherent point of } A \Longleftrightarrow \forall N \in \mathcal{N}(x), N \cap A \neq \emptyset. \tag{3.7}$$

The set of all adherent points to A *is called the closure of* A*, and it is denoted by* $Cl(A)$.

Remark 3.9. *It follows from this definition that* $A \subseteq Cl(A)$.

Example 3.17.

1. *If* A *is a subset of* $\mathbb{X}$ *with the indiscrete topology* $\mathcal{T}_{Ind}$*, then* $Cl(A) = \mathbb{X}$.
2. *If* A *is a subset of* $\mathbb{X}$ *with the discrete topology* $\mathcal{T}_{Disc}$*, then* $Cl(A) = A$.
3. *Consider the topological space* $(\mathbb{R}, |.|)$*, then:*
   - $\forall x \in \mathbb{R},\ Cl(\{x\}) = \{x\}$.
   - $\forall x, y \in \mathbb{R},\ Cl([x,y]) = Cl([x,y)) = Cl((x,y]) = Cl((x,y)) = [x,y]$.
   - $Cl(\mathbb{Q}) = Cl(\complement_{\mathbb{R}}\mathbb{Q}) = \mathbb{R}, \quad Cl(\mathbb{N}) = \mathbb{N}, \quad Cl(\mathbb{Z}) = \mathbb{Z}$.
4. *If* $\mathbb{X} = \{x, y, z, t\}$ *and* $\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{y\}, \{x, y\}\}$*, then for example:*
   - $Cl(\{x\}) = \{x, z, t\}, \quad Cl(\{y\}) = \{y, z, t\}$.
   - $Cl(\{z\}) = \{z, t\}, \quad Cl(\{t\}) = \{z, t\}$.

Proposition 3.14. $Cl(A)$ *is the smallest closed set that contains* A.

Proof. *We will show that $Cl(A)$ is the intersection of all closed sets that contain A. Let $F = \bigcap_{i \in I} F_i$, where F_i is a closed set that contains A for all $i \in I$.*

- $\left(F \overset{?}{\subseteq} Cl(A)\right)$ *Let $x \notin Cl(A)$. Then there exists an open set $O \in \mathcal{N}(x)$ such that $O \cap A = \emptyset$, which implies $A \subset \complement_{\mathbb{X}} O$. Therefore, $\complement_{\mathbb{X}} O$ is a closed set that contains A and $x \notin \complement_{\mathbb{X}} O$ which leads to $x \notin F$. Thus, we have:*

$$(i) \qquad F \subseteq Cl(A).$$

- $\left(Cl(A) \overset{?}{\subseteq} F\right)$ *Now, let $x \notin F$. Then $x \in \complement_{\mathbb{X}} F$ (which is open), but $\complement_{\mathbb{X}} F \cap A = \emptyset$, leading to $x \notin Cl(A)$. Thus, we have:*

$$(ii) \qquad Cl(A) \subseteq F.$$

From (i) and (ii), we conclude that $Cl(A) = F$. Therefore, $Cl(A)$ is the smallest closed set that contains A.

Remark 3.10. *The previous proposition allows us to write the following result:*

$$A \text{ is closed in } \mathbb{X} \Longleftrightarrow A = Cl(A). \tag{3.8}$$

Proposition 3.15. *Let A and B be two subsets of the topological space $(\mathbb{X}, \mathcal{T})$. Then, we have:*

1. $A \subseteq B \Longrightarrow Cl(A) \subseteq Cl(B)$.
2. $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
3. $Cl(A \cap B) \subset Cl(A) \cap Cl(B)$.
4. $\complement_{\mathbb{X}} Cl(A) = Int(\complement_{\mathbb{X}} A)$.
5. $Cl(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Int(A)$.
6. $Cl(Cl(A)) = Cl(A)$.

Proof. *(Exercise).*

Example 3.18. *If $A = (1,2)$ and $B = (2,3)$, then $Cl(A \cap B) = \emptyset$. However, $Cl(A) \cap Cl(B) = [1,2] \cap [2,3] = \{2\}$. This example shows that, in general, the inclusion in (3) is not an equality.*

3.5.3 Accumulation points

Definition 3.14. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space, $A \subset \mathbb{X}$, and $x \in \mathbb{X}$. We say that x is an accumulation point of A if and only if every $N \in \mathcal{N}(x)$ contains at least one point of A other than x. In other words:*

$$x \text{ is an accumulation point of } A \iff \forall N \in \mathcal{N}(x), (N \setminus \{x\}) \cap A \neq \emptyset. \tag{3.9}$$

The set of all accumulation points of A is called the derived set of A and is denoted by A'.

It follows from this definition that any point adherent to A but not belonging to A is an accumulation point. Therefore, we have the following result:

$$A' \cup A = Cl(A). \tag{3.10}$$

Example 3.19.

1. *Let $\mathbb{X} = \{x, y, z, t, s\}$, $\mathcal{T} = \{\emptyset, \mathbb{X}, \{x, y\}, \{z, t, s\}\}$, and $A = \{x, y, z\}$. Then, we have $A' = \{x, y, t, s\}$.*
2. *If A is a subset of a topological space $(\mathbb{X}, \mathcal{T}_{Disc})$, then $A' = \emptyset$.*
3. *In $(\mathbb{R}, |\cdot|)$, we have $\mathbb{N}' = \mathbb{Z}' = \emptyset$.*

Proposition 3.16. *A subset A of a topological space $(\mathbb{X}, \mathcal{T})$ is closed if and only if it contains all of its accumulation points.*

Proof. *Evident (from relation (3.10)).*

3.5.4 Isolated Points

Definition 3.15. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $A \subset \mathbb{X}$. We say that a point $x \in A$ is an isolated point if and only if there exists $N \in \mathcal{N}(x)$ such that N contains no other points of A except x. That is:*

$$x \text{ is an isolated point in } A \iff \exists N \in \mathcal{N}(x), \quad N \cap A = \{x\}.$$

The set of all isolated points of A is denoted by $Is(A)$.

Example 3.20.

1. *In the topological space* $(\mathbb{R}, |\cdot|)$, *we have* $Is(\mathbb{N}) = \mathbb{N}$ *and* $Is(\mathbb{Z}) = \mathbb{Z}$.
2. *Every point in a topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$ *is isolated.*
3. *Let* $\mathbb{X} = \{x, y, z, t, s\}$, $\mathcal{T} = \{\emptyset, \mathbb{X}, \{x\}, \{y\}, \{x, y\}\}$, *and* $A = \{y, z, t\}$. *Then,* $Is(A) = \{y\}$.

3.5.5 Boundary points

Definition 3.16. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$, *and* $x \in \mathbb{X}$. *We say that* x *is a boundary point of* A *if it adheres to both* A *and* $\complement_{\mathbb{X}} A$. *In other words:*

$$x \text{ is a boundary point of } A \iff x \in Cl(A) \cap Cl(\complement_{\mathbb{X}} A).$$

The set of all boundary points of A *is called the boundary of* A *and is denoted by* $\partial(A)$.

Remark 3.11. *Using property (5) of Proposition (3.15), we obtain:*

$$\begin{aligned} \partial(A) &= Cl(A) \cap Cl(\complement_{\mathbb{X}} A) \\ &= Cl(A) \cap \complement_{\mathbb{X}} Int(A) \\ &= Cl(A) - Int(A). \end{aligned} \tag{3.11}$$

Proposition 3.17. *Let* A *be a subset of a topological space* $(\mathbb{X}, \mathcal{T})$. *Then,*

1. $\partial(A)$ *is a closed set.*
2. A *is both open and closed* $\iff \partial(A) = \emptyset$.
3. A *is open* $\iff \partial(A) \cap A = \emptyset$.
4. A *is closed* $\iff \partial(A) \subseteq A$.

Proof. *(Exercise).*

Example 3.21.

1. *If* A *is a subset of a topological space* $(\mathbb{X}, \mathcal{T}_{Disc})$, *then* $\partial(A) = \emptyset$.
2. *In the space* $(\mathbb{R}, |\cdot|)$:

- *If* $A = (a,b)$, *then* $\partial(A) = Cl(A) - Int(A) = [a,b] - (a,b) = \{a,b\}$.
- *If* $A = \mathbb{Z}$, *then* $\partial(A) = Cl(A) - Int(A) = \mathbb{Z} - \emptyset = \mathbb{Z}$.

3.5.6 Exterior points

Definition 3.17. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space,* $A \subset \mathbb{X}$*, and* $x \in \mathbb{X}$*. We say that* x *is an exterior point of* A *if it belongs to the interior of* $\complement_{\mathbb{X}} A$*. In other words:*

$$x \text{ is an exterior point of } A \iff x \in Int(\complement_{\mathbb{X}} A).$$

The set of all exterior points of A *is called the exterior of* A*, and it is denoted by* $Ext(A)$.

Remark 3.12. *Using property (4) from Proposition (3.15), we obtain the following result:*

$$Ext(A) = Int(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Cl(A).$$

Proposition 3.18. *Let* A *and* B *be two subsets of a topological space* $(\mathbb{X}, \mathcal{T})$*. Then,*

1. $Ext(A)$ *is an open set.*
2. $Ext(A) \subseteq \complement_{\mathbb{X}} A$.
3. $Ext(A) = Ext(\complement_{\mathbb{X}} Ext(A))$.
4. $Ext(A \cup B) = Ext(A) \cap Ext(B)$.
5. $Cl(A) = \mathbb{X} \iff Ext(A) = \emptyset$.

Proof. *(Exercise).*

3.5.7 Dense sets

Definition 3.18. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space, and let* A *and* B *be two subsets of* $\mathbb{X}$*. We say that* A *is dense in* B *if and only if every point of* B *is an adherent point of* A*, in other words:*

$$A \text{ is dense in } B \iff B \subseteq Cl(A), \tag{3.12}$$

and we say that A is dense in $\mathbb{X}$ if and only if $Cl(A)=\mathbb{X}$ or $Int(\complement_{\mathbb{X}}A)=\emptyset$.

Example 3.22.

1. *If $\mathbb{X}$ is equipped with the indiscrete topology, then every non-empty subset of $\mathbb{X}$ is dense in $\mathbb{X}$.*

2. *If $\mathbb{X}$ is equipped with the discrete topology, and A and B are subsets of $\mathbb{X}$ such that $B\subset A$, then A is dense in B. Moreover, no subset $A\neq\mathbb{X}$ is dense in $\mathbb{X}$.*

3. *In $(\mathbb{R},|\cdot|)$, let $A=[a,b)$ and $B=(a,b)$. It is clear that A is dense in B because $B\subseteq Cl(A)=[a,b]$.*

4. *We have seen that $\mathbb{Q}$ is dense in $\mathbb{R}$ since $Cl(\mathbb{Q})=\mathbb{R}$.*

5. *Let $\mathbb{X}=\{x,y,z,t\}$ and $\mathcal{T}=\{\emptyset,\mathbb{X},\{x\},\{x,y\}\}$. Define $A=\{t\}$ and $B=\{x,z\}$; we find that B is dense in A because $A\subseteq Cl(B)=\mathbb{X}$, but A is not dense in B since $B\nsubseteq Cl(A)=\{z,t\}$.*

Proposition 3.19. *Let $(\mathbb{X},\mathcal{T})$ be a topological space, and consider three subsets A, B, and C of $\mathbb{X}$. If A is dense in B and B is dense in C; then, A is dense in C.*

Proof. *On the one hand, since A is dense in B, we have $B\subseteq Cl(A))$, which implies that*

$$Cl(B)\subseteq Cl(A). \tag{i}$$

On the other hand, since B is dense in C, we have

$$C\subseteq Cl(B). \tag{ii}$$

From (i) and (ii), we conclude that $C\subseteq Cl(A)$, so A is dense in C.

Remark 3.13. *The previous proposition shows that density is a transitive property.*

The following property is a very practical characterization of dense subsets.

Proposition 3.20. *Let $(\mathbb{X},\mathcal{T})$ be a metric space, and let $A\subseteq\mathbb{X}$. Then, A is dense in $\mathbb{X}$ if and only if every non-empty open set in $\mathbb{X}$ contains at least one element of A.*

Proof.

$\Longrightarrow$) *Suppose that A is a dense subset of $\mathbb{X}$ and O is a non-empty open set in $\mathbb{X}$. Since $Cl(A) = \mathbb{X}$, it follows that $O \subseteq Cl(A)$. Thus, $A \cap O \neq \emptyset$ because O is a neighborhood of each of its points.*

$\Longleftarrow$) *Assume that $A \cap O \neq \emptyset$ for every open set O in $\mathbb{X}$. This implies that for any neighborhood N of any point $x \in \mathbb{X}$, we also have $N \cap A \neq \emptyset$, since N contains a non-empty open set. Therefore, $x \in Cl(A)$, and consequently, $Cl(A) = \mathbb{X}$.*

3.6 Separated Spaces (Hausdorff Spaces))

Definition 3.19. *A topological space $(\mathbb{X}, \mathcal{T})$ is said to be separated or Hausdorff if and only if, for any two distinct points x and y in $\mathbb{X}$, there exist neighborhoods $N \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $N \cap W = \emptyset$.*

Example 3.23.

1. *The space $(\mathbb{X}, \mathcal{T}_{Disc})$ is separated.*
2. *If $\mathrm{card}(\mathbb{X}) \geq 2$, the space $(\mathbb{X}, \mathcal{T}_{Ind})$ is not separated.*
3. *The metric space $(\mathbb{R}, |.|)$ is Hausdorff.*
4. *The space $(\mathbb{X}, \mathcal{T}_{Cof})$ is not separated.*

Proposition 3.21. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Then, $\mathbb{X}$ is separated if and only if for every $x \in \mathbb{X}$, we have $\{x\} = \bigcap_{N_x \in \mathcal{N}(x)} N_x$, where N_x is a closed neighborhood of x.*

Proof.

$\Longrightarrow$) *Let $\mathbb{X}$ be a separated topological space and $x \in \mathbb{X}$. We want to show that*

$$\{x\} = \bigcap_{N_x \in \mathcal{N}(x)} N_x, \qquad \text{(i)}$$

where N_x is a closed neighborhood of x. Suppose there exists $y \in \bigcap_{N_x \in \mathcal{N}(x)} N_x$ such that $y \neq x$. Then, there exist two open neighborhoods U and W of x and y, respectively, such that $U \cap W = \emptyset$. This means that $C_{\mathbb{X}}W$ is a closed neighborhood of x (since it contains U), which contradicts the fact that y belongs to all closed neighborhoods of x.

$\Longleftarrow$) *Conversely, let $x, y \in \mathbb{X}$ such that $x \neq y$. From the equality (i), it follows that there exists a closed neighborhood N_x of x that does not contain y. Therefore, there exists an open set O such that $x \in O \subset Cl(O) \subset N_x$, which implies that $y \notin Cl(O)$. Finally, we conclude that O and $C_{\mathbb{X}} Cl(O)$ are two disjoint open sets containing x and y, respectively, which shows that $\mathbb{X}$ is a separated space.*

Using the previous proposition, we obtain the following result:

Proposition 3.22. *Every singleton in a separated space is closed, and in general, every finite set in a separated space is closed.*

Proposition 3.23. *Let $(\mathbb{X}, \mathcal{T})$ be a separated topological space and $x \in \mathbb{X}$. Then, x is an accumulation point of a subset A of $\mathbb{X}$ if and only if every neighborhood N_x of x contains infinitely many elements of A.*

Proof.

$\Longleftarrow$) *Obvious.*

$\Longrightarrow$) *Suppose there exists a neighborhood $N_x \in \mathcal{N}(x)$ that contains a finite number of elements $\{x_1, x_2, \ldots, x_n\}$ of A. Then, $W = N_x \setminus \{x_1, x_2, \ldots, x_n\}$ is a neighborhood of x and $(W \setminus \{x\}) \cap A = \emptyset$. Therefore, x is not an accumulation point of A.*

Remark 3.14. *It follows from the previous proposition that any finite subset of a separated topological space has no accumulation points.*

3.7 Induced topology, Product topology

3.7.1 Induced topology

Definition 3.20. *Let $(\mathbb{X}, \mathcal{T})$ a topological space and A a subset of $\mathbb{X}$. Then,*

$$\mathcal{T}_A = \{O_A = A \cap O : O \in \mathcal{T}\}, \tag{3.13}$$

is a topology in A. The open sets in A are the intersections of open sets in $\mathbb{X}$ with A. This topology is called the induced topology or relative topology of A in $\mathbb{X}$, and $(A, \mathcal{T})$ is *called a topological subspace of* $(\mathbb{X}, \mathcal{T})$.

Exercise. *Show that* $\mathcal{T}_A$ *is a topology on* A.

Example 3.24. *Consider the following topology*

$$\mathcal{T} = \{\mathbb{X}, \emptyset, \{x\}, \{z,t\}, \{x,z,t\}, \{y,z,t,s\}\}$$

on $\mathbb{X} = \{x,y,z,t,e\}$ *and the subset* $A = \{x,t,s\}$ *of* $\mathbb{X}$. *Then we have:* $\mathbb{X} \cap A = A$, $\emptyset \cap A = \emptyset$, $\{x\} \cap A = \{x\}$, $\{z,t\} \cap A = \{t\}$, $\{x,z,t\} \cap A = \{x,t\}$, *and* $\{y,z,t,s\} \cap A = \{t,s\}$. *Thus, the topology induced by* $\mathcal{T}$ *on* A *is*

$$\mathcal{T}_A = \{A, \emptyset, \{x\}, \{t\}, \{x,t\}, \{t,s\}\}.$$

Example 3.25. *Consider the usual topology on* $\mathbb{R}$ *and the induced topology* $\mathcal{T}_A$ *on the closed interval* $A = [4,9]$. *Note that the half-open interval* $[4,6[$ *is an open set in the topology* $\mathcal{T}_A$ *because* $[4,6[=]3,6[\cap A$, *where* $]3,6[$ *is an open set in* $\mathbb{R}$. *Thus, we see that a set can be open relative to a subspace but neither open nor closed in the entire space.*

Example 3.26. *Consider the usual topology on* $\mathbb{R}$ *and the induced topology* $\mathcal{T}_A$ *on* $A = \mathbb{N}$. *Then, for any* $n \in \mathbb{N}$, *we have* $\mathbb{N} \cap (n-1, n+1) = \{n\} \in \mathcal{T}_A$. *We conclude that* $(\mathbb{N}, \mathcal{T}_{\mathbb{N}} = \mathcal{P}(\mathbb{N}))$ *is a discrete space.*

Proposition 3.24. *Let* $(A, \mathcal{T}_A)$ *be a subspace of a topological space* $(\mathbb{X}, \mathcal{T})$, *and let* F' *be a subset of* A. *Then,* F' *is closed in* A *with respect to the induced topology* $\mathcal{T}_A$ *if and only if there exists* $F \in \mathcal{F}$ *(where* $\mathcal{F}$ *is the set of closed sets in* $\mathbb{X}$*) such that* $F' = A \cap F$.

Proof. *We have that* F' *is closed in* A *if and only if* $\complement_A F'$ *is open in* A, *i.e., if and only if there exists* $O \in \mathcal{T}$ *such that* $\complement_A F' = A \cap O$. *Therefore,* F' *is closed in* A *if and only if there exists* $O \in \mathcal{T}$ *such that*

$$F' = \complement_A(\complement_A F') = \complement_A(A \cap O) = A \cap (\complement_{\mathbb{X}} O),$$

i.e., if and only if there exists $F = \complement_{\mathbb{X}} O \in \mathcal{F}$ *such that* $F' = A \cap F$.

We can easily show the following result.

Proposition 3.25. *Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(\mathbb{X}, \mathcal{T})$, and let B be a subset of A. If B is open (resp. closed) in $\mathbb{X}$, then B is open (resp. closed) in A.*

Proof. *It suffices to see that $B = B \cap A$.*

Remark 3.15. *The two examples (3.25) and (3.26) show that the converse of the previous result is not necessarily true.*

Proposition 3.26. *Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(\mathbb{X}, \mathcal{T})$. Then every open (resp. closed) set in A is an open (resp. closed) set in $\mathbb{X}$ if and only if A is an open (resp. closed) set in $\mathbb{X}$.*

Proof.

$\Longrightarrow$) *Suppose that every open set in A is an open set in $\mathbb{X}$, then A is an open set in $\mathbb{X}$.*

$\Longleftarrow$) *Suppose that A is an open set in $\mathbb{X}$ and let O_A be an open set in A. Then there exists $O \in \mathcal{T}$ such that $O_A = A \cap O$, which is an open set in $\mathbb{X}$ since $A \in \mathcal{T}$.*
By similar arguments, this result can be shown for closed sets.

Proposition 3.27.

1. *If $x \in A$, then N' is a neighborhood of x in A if and only if there exists $N \in \mathcal{N}(x)$ such that $N' = N \cap A$.*
2. *If $\mathcal{S}(x)$ is a neighborhood base of x in $\mathbb{X}$, then $\{N \cap A : N \in \mathcal{S}(x)\}$ is neighborhood base of x in A for the induced topology $\mathcal{T}_A$.*
3. *If B is a subset of A, then we have:*
   - ***a**) $Cl(B)_A = A \cap Cl(B)$ (where $Cl(B)_A$ and $Cl(B)$ are the closures of B for $\mathcal{T}_A$ and $\mathcal{T}$, respectively).*
   - ***b**) $Cl(B)_A = Cl(B) \Longleftrightarrow A$ is closed in $\mathbb{X}$.*
   - ***c**) $A \cap Int(B) \subset Int(B)_A$*
4. *If $\mathfrak{B}$ is a base for $(\mathbb{X}, \mathcal{T})$, then $\mathfrak{B}_A = \{\beta \cap A : \beta \in \mathfrak{B}\}$ is a base for $(A, \mathcal{T}_A)$.*

Proof.

1. *If N' is a neighborhood of x in A, then there exists an open set $A \cap O \in \mathcal{T}_A$ (i.e., there exists $O \in \mathcal{T}$) such that $x \in A \cap O \subset N'$. Thus, if we define $N = O \cup N'$, we obtain $x \in O \subset N$, so N is a neighborhood of x in $\mathbb{X}$, and we have:*

$$N \cap A = (O \cup N') \cap A = (O \cap A) \cup (N' \cap A) = (O \cap A) \cup N' = N'.$$

Conversely, if $N \in \mathcal{N}(x)$, then there exists $O \in \mathcal{T}$ such that $x \in O \subset N$. Thus, $x \in A \cap O \subset A \cap N$, and therefore $N' = A \cap N$ is a neighborhood of x in A because $A \cap O$ is open in A.

2. *Let $N' = N \cap A$ be a neighborhood of x in A for the induced topology $\mathcal{T}_A$, with N being a neighborhood of x in $\mathbb{X}$. If $\mathcal{S}(x)$ is a neighborhood base of x in $\mathbb{X}$, then there exists $W \in \mathcal{S}(x)$ such that $W \subset N$, so $W \cap A \subset N'$. This leads to the conclusion that $\{N \cap A : N \in \mathcal{S}(x)\}$ is neighborhood base of x in A.*

3. ***a**) If $x \in Cl(B)_A$, then for every $N \in \mathcal{N}(x)$ (for the topology $\mathcal{T}$), we have $(N \cap A) \cap B \neq \emptyset$, and therefore $x \in A$ and $x \in Cl(B)$, from which we obtain*

   (i) $$x \in A \cap Cl(B).$$

   On the other hand, if $x \in A \cap Cl(B)$, then every neighborhood $N \cap A$ of x in A intersects B because N intersects B and $B \subset A$, from which we obtain

   (ii) $$x \in Cl(B)_A.$$

   Finally, from (i) and (ii), we conclude that $Cl(B)_A = A \cap Cl(B)$.

   ***b**) Suppose that for every subset B of A, we have $Cl(B)_A = Cl(B)$, then $A = Cl(A)_A = Cl(A)$ because A is closed in A, hence A is closed in $\mathbb{X}$.*

   Conversely, if A is closed in $\mathbb{X}$, then $Cl(B) \subset Cl(A) = A$, and thus $Cl(B)_A = A \cap Cl(B) = Cl(B)$.

   ***c**) We have $A \cap Int(B)$ is an open set in A contained in B, so $A \cap Int(B) \subset Int(B)_A$.*

4. *Let U be an open set of A, then there exists $O \in \mathcal{T}$ such that $U = A \cap O$, but $O = \bigcup_{i \in I} \beta_i$, where $\beta_i \in \mathfrak{B}$ for all $i \in I$, from which we obtain*

$$U = A \cap \left(\bigcup_{i \in I} \beta_i \right) = \bigcup_{i \in I} (A \cap \beta_i),$$

which completes the proof.

Definition 3.21. *A topological property is hereditary if whenever a topological space possesses this property, it also holds for each of its sub-spaces.*

Proposition 3.28. *Every subspace of a separated space is separated.*

Proof. *Let $(A, \mathcal{T}_A)$ be a topological subspace of a separated topological space $(\mathbb{X}, \mathcal{T})$, and let $x, y \in A$ such that $x \neq y$. Since $\mathbb{X}$ is separated, there exist neighborhoods $N \in \mathcal{N}(x)$ and $W \in \mathcal{N}(y)$ such that $N \cap W = \emptyset$, hence $(A \cap N) \cap (A \cap W) = \emptyset$. Therefore, $(A \cap N)$ and $(A \cap W)$ are disjoint neighborhoods of x and y, respectively, within A, which shows that A is separated.*

The following result shows the transitivity of the induced topology.

Proposition 3.29. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $B \subset A \subset \mathbb{X}$ two subsets of $\mathbb{X}$. We denote by $\mathcal{T}'_B$ the topology induced on B by $\mathcal{T}_A$. Then, we have*

$$\mathcal{T}_B = \mathcal{T}'_B.$$

Proof. *If $U \in \mathcal{T}_B$, then there exists $O \in \mathcal{T}$ such that $U = B \cap O$, and since $A \cap O \in \mathcal{T}_A$, we obtain $U = B \cap O = B \cap (A \cap O) \in \mathcal{T}'_B$.*

Conversely, if $U \in \mathcal{T}'_B$, then there exists $O_A \in \mathcal{T}_A$ such that $U = B \cap O_A$, and since $O_A \in \mathcal{T}_A$, there exists $O \in \mathcal{T}$ such that $O_A = A \cap O$. Thus, $U = B \cap (A \cap O) = B \cap O$, and therefore $U \in \mathcal{T}_B$.

3.7.2 Product topology

Definition 3.22. *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i = 1, \dots, n\}$ be a collection of topological spaces. The box topology or product topology on the product $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$ is the topology with basis*

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} O_i \ : \ O_i \in \mathcal{T}_i \text{ for each } 1 \leqslant i \leqslant n \right\}.$$

So we can always make the product of topological space into a topological space using the box topology.

Proof.

1. *We have* $\mathbb{X}=\mathbb{X}_1\times\mathbb{X}_2\times\cdots\times\mathbb{X}_n\in\mathcal{T}$ *and* $\underbrace{\emptyset\times\emptyset\times\cdots\times\emptyset}_{n \text{ times}}\in\mathcal{T}$ *because they are elements of* $\mathcal{B}$.

2. *If* $\{O_i : i\in I\}$ *is a family of open subsets of* $\mathbb{X}$, *then we have:*

$$\bigcup_{i\in I}O_i=\bigcup_{i\in I}\left(\bigcup_{j\in J}\left(O^1_{i,j}\times O^2_{i,j}\times\cdots\times O^n_{i,j}\right)\right)=\bigcup_{(i,j)\in I\times J}\left(O^1_{i,j}\times O^2_{i,j}\times\cdots\times O^n_{i,j}\right)\in\mathcal{T},$$

because $O^1_{i,j}\times O^2_{i,j}\times\cdots\times O^n_{i,j}\in\mathcal{B}$ *for all* $i\in I$ *and* $j\in J$.

3. *It suffices to show that if* $O_1,O_2\in\mathcal{T}$, *then* $O_1\cap O_2\in\mathcal{T}$. *Since* $O_1,O_2\in\mathcal{T}$, *we have* $O_1=\underset{i\in I}{\cup}N_i$ *and* $O_2=\underset{j\in J}{\cup}W_j$ *where* N_i, $W_j\in\mathcal{B}$ *for all* $i\in I$ *and* $j\in J$. *Therefore, we obtain:*

$$O_1\cap O_2=\left(\underset{i\in I}{\cup}N_i\right)\cap\left(\underset{j\in J}{\cup}W_j\right)=\underset{(i,j)\in I\times J}{\cup}(N_i\cap W_j).$$

It remains to show that $N_i\cap W_j\in\mathcal{B}$ *for all* $i\in I$ *and* $j\in J$. *By definition, we have* $N_i=R^i_1\times\cdots\times R^i_n$ *and* $W_j=K^j_1\times\cdots\times K^j_n$ *where* $R^i_\alpha\in\mathcal{T}_\alpha$ *and* $K^j_\alpha\in\mathcal{T}_\alpha$ *for all* $\alpha=1,\ldots,n$. *This allows us to write:*

$$N_i\cap W_j=(R^i_1\cap K^j_1)\times(R^i_2\cap K^j_2)\times\cdots\times(R^i_n\cap K^j_n).$$

Since $R^i_\alpha\cap K^j_\alpha$ *are open sets in* $\mathcal{T}_\alpha$ *for all* $\alpha=1,\ldots,n$, *we deduce that* $N_i\cap W_j\in\mathcal{B}$ *for all* $i\in I$ *and* $j\in J$, *which implies that* $O_1\cap O_2\in\mathcal{T}$. *Finally, we conclude that* $\mathcal{T}$ *is a topology on* $\mathbb{X}$.

Example 3.27.

1. *The box topology or product topology on* $\mathbb{R}^n$, *such that* $\mathbb{R}$ *is equipped with the usual topology, is the topology with basis*

$$\mathcal{B}=\left\{\prod_{i=1}^{n}]a_i,b_i[\ :\ a_i,b_i\in\mathbb{R}\text{ for each }1\leqslant i\leqslant n\right\}.$$

2. *Let* $\{(\mathbb{X}_i,\mathcal{T}_i) : i=1,\ldots,n\}$ *be a family of indiscrete spaces. Then, the product* $\mathbb{X}=\prod_{i=1}^{n}\mathbb{X}_i$ *is an indiscrete space. Indeed, if* $O=\prod_{i=1}^{n}O_i\neq\mathbb{X}$, *then there exists an index* i_0 *such that* $O_{i_0}\neq\mathbb{X}_{i_0}$. *Since* $\mathcal{T}_{i_0}=\{\mathbb{X}_{i_0},\emptyset\}$, *we obtain* $O_{i_0}=\emptyset$, *and hence* $O=\emptyset$. *Therefore, the family* $\{\mathbb{X},\emptyset\}$ *forms a basis for the product topology on* $\mathbb{X}$, *which shows that* $\mathbb{X}$ *is an indiscrete space.*

Proposition 3.30. *Let* $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$ *be a product of topological spaces, and let* $x = (x_1, \ldots, x_n) \in \mathbb{X}$. *Let* $\mathcal{S}$ *denote the family of sets of the form* $N_1 \times \cdots \times N_n$, *where* $N_i \in \mathcal{N}(x_i)$ *in* $\mathbb{X}_i$ *for* $i = 1, \ldots, n$. *Then,* $\mathcal{S}$ *is a basic neighborhoods of* x *in* $\mathbb{X}$.

Proof.

If $N_i \in \mathcal{N}(x_i)$, *then there exists* $O_i \in \mathcal{T}_i$, *for all* $i = 1, \ldots, n$, *such that* $x_i \in O_i \subset V_i$. *Therefore, we obtain* $x \in O_1 \times \cdots \times O_n \subset N_1 \times \cdots \times N_n$, *and since* $O_1 \times \cdots \times O_n$ *is an open set in* $\mathbb{X}$, *we conclude that* $N_1 \times \cdots \times N_n$ *is a neighborhood of* x *in* $\mathbb{X}$.

Now, let $N \in \mathcal{N}(x)$ *in* $\mathbb{X}$. *Then, there exists an open set* $O \subset \mathbb{X}$ *such that* $x \in O \subset N$. *Thus, there exists* $W = O_1 \times \cdots \times O_n$ *an open set containing* x *(since* $\mathcal{B}$ *is a basis for the product topology on* $\mathbb{X}$ *(see Definition 3.22)). Hence,* $W \in \mathcal{S}$ *because* $O_i \in \mathcal{N}(x_i)$ *for all* $i = 1, \ldots, n$, *which implies that* $W \subset N$.

Example 3.28. *Let* $\mathbb{R}^n$ *be equipped with the usual topology, and let* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. *The family*

$$\left\{ \prod_{i=1}^{n} (x_i - \varepsilon_i, x_i + \varepsilon_i) \ : (\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{R}_+^*)^n \right\},$$

is a basic neighborhoods of x. *Similarly, the family*

$$\left\{ \prod_{i=1}^{n} (x_i - \varepsilon, x_i + \varepsilon) \ : \varepsilon \in \mathbb{R}_+^* \right\},$$

is also a basic neighborhoods of x.

Proposition 3.31. *Consider* $A = \prod_{i=1}^{n} A_i$, *a subset of the product space* $\mathbb{X} = \prod_{i=1}^{n} \mathbb{X}_i$. *The closure of* A, *denoted by* $Cl(A)$, *is given by:*

$$Cl(A) = \prod_{i=1}^{n} Cl(A_i).$$

Proof. *Let* $x = (x_1, \ldots, x_n) \in \mathrm{Cl}(A)$. *Then, for every* $N_i \in \mathcal{N}(x_i)$, *we have:*

$$(N_1 \cap A_1) \times \cdots \times (N_n \cap A_n) = (N_1 \times \cdots \times N_n) \cap A \neq \emptyset,$$

which implies $N_i \cap A_i \neq \emptyset$ *for all* $i = 1, \ldots, n$. *Thus,* $x_i \in Cl(A_i)$ *for all* $i = 1, \ldots, n$, *showing that* $x \in \prod_{i=1}^{n} Cl(A_i)$.

Conversely, if $x \in \prod\limits_{i=1}^{n} Cl(A_i)$*, then for every* $N_i \in \mathcal{N}(x_i)$*,* $i = 1, \ldots, n$*, we have* $N_i \cap A_i \neq \emptyset$*. Therefore,*

$$(N_1 \cap A_1) \times \cdots \times (N_n \cap A_n) = (V_1 \times \cdots \times N_n) \cap A \neq \emptyset,$$

which shows that $x \in Cl(A)$*.*

Using the previous proposition, we obtain the following result.

Proposition 3.32. *Let* $A = \prod\limits_{i=1}^{n} A_i$ *be a subset of a product space* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$*. Then* A *is closed in* $\mathbb{X}$ *if and only if* A_i *is closed in* $\mathbb{X}_i$ *for every* $i = 1, \ldots, n$*.*

Proposition 3.33. *A product space* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *is Hausdorff if and only if each* $\mathbb{X}_i$ *is Hausdorff for every* $i = 1, \ldots, n$*.*

Proof.

$\Longrightarrow$) *Suppose that* $\mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *is Hausdorff, and let* $x_{i_0}, y_{i_0} \in \mathbb{X}_{i_0}$ *such that* $x_{i_0} \neq y_{i_0}$*. For any* $x' = (x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_n) \in \prod\limits_{\substack{i=1 \\ i \neq i_0}}^{n} \mathbb{X}_i$*, there exists a neighborhood* O *of* $(x_1, \ldots, x_{i_0}, \ldots, x_n)$ *and a neighborhood* O' *of* $(x_1, \ldots, y_{i_0}, \ldots, x_n)$ *such that* $O \cap O' = \emptyset$*. Let* $O = N_1 \times N_2$ *and* $O' = N_1' \times N_2'$*, where* $N_1 \in \mathcal{N}(x_{i_0})$*,* $N_2 \in \mathcal{N}(x')$*,* $N_1' \in \mathcal{N}(y_{i_0})$*, and* $N_2' \in \mathcal{N}(x')$*. Thus, we obtain:*

$$O \cap O' = (N_1 \cap N_1') \times (N_2 \cap N_2') = \emptyset \Longrightarrow N_1 \cap N_1' = \emptyset,$$

and therefore $\mathbb{X}_{i_0}$ *is Hausdorff.*

$\Longleftarrow$) *Let* $x = (x_1, \ldots, x_n)$*,* $y = (y_1, \ldots, y_n) \in \mathbb{X} = \prod\limits_{i=1}^{n} \mathbb{X}_i$ *such that* $x \neq y$*. Then there exists at least one* $i_0 \in \{1, \ldots, n\}$ *such that* $x_{i_0} \neq y_{i_0}$*. Since* $\mathbb{X}_{i_0}$ *is Hausdorff, there exist a neighborhood* N *of* x_{i_0} *and a neighborhood* W *of* y_{i_0} *such that* $V \cap W = \emptyset$*. By setting* $O_x = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i_0-1} \times N \times \mathbb{X}_{i_0+1} \times \cdots \times \mathbb{X}_n$ *and* $O_y = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i_0-1} \times W \times \mathbb{X}_{i_0+1} \times \cdots \times \mathbb{X}_n$*, we obtain* $O_x \in \mathcal{N}(x)$*,* $O_y \in \mathcal{N}(y)$*, and* $O_x \cap O_y = \emptyset$*, which shows that* $\mathbb{X}$ *is Hausdorff.*

3.8 Convergent sequences

Definition 3.23. *A "sequence of elements" of a set $\mathbb{X}$ is defined as any function from $\mathbb{N}$ (or a subset of $\mathbb{N}$) into $\mathbb{X}$, which associates with each integer n in $\mathbb{N}$ an element of $\mathbb{X}$ denoted by x_n. The sequence with general term x_n is denoted by $(x_n)_{n\in\mathbb{N}}$.*

Definition 3.24. *Let $(\mathbb{X},\mathcal{T})$ be a topological space. Consider a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $\mathbb{X}$ and a point $l \in \mathbb{X}$. We say that l is the limit of the sequence $(x_n)_{n\in\mathbb{N}}$ (or that $(x_n)_{n\in\mathbb{N}}$ converges to l) as n tends to infinity, if for every neighborhood N of l in $\mathbb{X}$, there exists an integer n_0 such that $x_n \in N$ for all $n \geq n_0$. In other words,*

$$\forall N \in \mathcal{N}(l),\ \exists n_0 \in \mathbb{N} \ :\ \forall n \in \mathbb{N},\ n \geqslant n_0 \Rightarrow x_n \in N.$$

In this case, we write:

$$\lim_{n\to\infty} x_n = l.$$

A sequence that does not converge is called divergent.

Example 3.29.

1. *Every constant sequence is convergent in all topological spaces.*
2. *A sequence in an indiscrete space is convergent to every point of that space.*
3. *If $(\mathbb{X},\mathcal{T})$ is a discrete space, then a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{X}$ converges to l if and only if there exists n_0 such that $x_n = l$ for all $n \geqslant n_0$.*
4. *The sequence (x_n) of the general term $x_n = \dfrac{1}{n}$ is convergent to 0 in $(\mathbb{R}, |\cdot|)$, and it is divergent in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$.*

Proposition 3.34. *If $(\mathbb{X},\mathcal{T})$ is a Hausdorff topological space, then every convergent sequence in $\mathbb{X}$ has a unique limit.*

Proof. *Let us reason by contradiction. Let (x_n) be a convergent sequence in $\mathbb{X}$. Suppose it has two distinct limits $l_1 \neq l_2$. Since $(\mathbb{X},\mathcal{T})$ is a Hausdorff space, there exist neighborhoods $N_1 \in \mathcal{N}(l_1)$ and $N_2 \in \mathcal{N}(l_2)$ such that $N_1 \cap N_2 = \emptyset$. According to the definition (3.24), there exist integers n_1 and n_2 such that:*

$$\forall n \geqslant n_1, x_n \in N_1 \quad \text{and} \quad \forall n \geqslant n_2, x_n \in N_2.$$

Let $n_0 = \max(n_1, n_2)$. *Then, for all* $n \geqslant n_0$, *we have*

$$x_n \in N_1 \cap N_2,$$

which contradicts the fact that $N_1 \cap N_2 = \emptyset$. *Therefore,* $l_1 = l_2$.

Example 3.30. *The trivial topology (indiscrete topology) on a set* $\mathbb{X}$ *is a non-Hausdorff topology because every element* $x \in \mathbb{X}$ *has only one neighborhood, namely* $\mathbb{X}$ *itself. Therefore, if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathbb{X}$, *every point* $x \in \mathbb{X}$ *is a limit for this sequence. Hence, the limit is not unique.*

Definition 3.25. *A cluster point or accumulation point of a sequence* $(x_n)_{n\in\mathbb{N}}$ *in a topological space* $(\mathbb{X}, \mathcal{T})$ *is a point* x *such that, for every neighborhood* N *of* x, *there are infinitely many natural numbers* n *such that* $x_n \in N$.

Remark 3.16. *According to the previous definition, we conclude that the limit of a sequence is an accumulation (cluster point) point of this sequence.*

Example 3.31.

1. *In* $(\mathbb{R}, |.|)$, $x = 1$ *is the unique accumulation point (cluster point) of the sequence* $(x_n)_{n\in\mathbb{N}} = (1 + e^{-n})_{n\in\mathbb{N}}$, *and this value is the limit of the sequence. Moreover,* $x_n = 1 + e^{-n}$ *is an adherent point for every* $n \in \mathbb{N}$, *but it is not an accumulation point (cluster point).*

2. *In* $(\mathbb{R}, |.|)$, *the sequence* $(x_n)_{n\in\mathbb{N}} = ((-1)^n)_{n\in\mathbb{N}}$ *has two accumulation points (cluster points),* -1 *and* 1, *but it is a divergent sequence.*

According to the previous example and the definition (3.25), we conclude that every accumulation point is an adherent point, but the converse is not true.

Proposition 3.35. *If* $(\mathbb{X}, \mathcal{T})$ *is a Hausdorff (separated) topological space, then every convergent sequence in* $\mathbb{X}$ *has a unique accumulation point (cluster points), which is its limit.*

Proof. *By arguments similar to those used in the proof of the previous proposition.*

Remark 3.17.

1. *A sequence that has at least two accumulation points diverges.*

2. *The converse of the previous proposition is false. For example, the sequence defined by* $x_n = (1-(-1)^n) \times n$ *has only* 0 *as an accumulation point but diverges.*

Definition 3.26. *Let* (x_n) *be a sequence in a topological space* $(\mathbb{X}, \mathcal{T})$. *We call a subsequence or extracted sequence of* (x_n) *any sequence of the form* $(x_{\phi(n)})$, *where* $\phi(n)$ *is a strictly increasing function from* $\mathbb{N}$ *to* $\mathbb{N}$.

Example 3.32. *If* (x_n) *is a sequence in a topological space* $(\mathbb{X}, \mathcal{T})$ *and* $\phi(n) = 2n+1$, *then* $(x_{2n+1}) = \{x_1, x_3, x_5, x_7, \ldots, x_{2n+1}, \ldots\}$ *is a subsequence of* (x_n).

Using the definitions (3.24) and (3.25), we obtain the following two results.

Proposition 3.36.

1. *Every subsequence of a convergent sequence is convergent (towards the same limit).*

2. *The limit of a subsequence extracted from a sequence* (x_n) *is a cluster point of this sequence.*

Proposition 3.37. *Let* $(z_n) = \{z_n^1, z_n^2, \ldots, z_n^k\}$ *be a sequence in a space* $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$. *Then,* (z_n) *converges to* $z = (z^1, z^2, \ldots, z^k)$ *if and only if for all* $i = 1, \ldots, k$, *the sequence* (z_n^i) *converges in* $\mathbb{X}$ *to* z^i.

Proof.

$\Longrightarrow$) *Suppose that* $(z_n) = \{z_n^1, z_n^2, \ldots, z_n^k\}$ *converges in* $\mathbb{X}$ *to* $z = (z^1, z^2, \ldots, z^k)$. *Let* N_i *be a neighborhood of* z_i *in* $\mathbb{X}_i$, *for* $i = 1, \ldots, k$. *Then,* $W = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times N_i \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_k$ *is a neighborhood of* z, *so there exists* $n_0 \in \mathbb{N}$ *such that*

$$n \geqslant n_0 \Longrightarrow z_n \in W.$$

Consequently, we obtain:

$$n \geqslant n_0 \Longrightarrow z_n^i \in N_i,$$

which shows that for all $i = 1, \ldots, k$, *the sequence* (z_n^i) *converges to* z^i *in* $\mathbb{X}_i$.

$\Longleftarrow$) *Suppose that for all* $i = 1, \ldots, k$, *the sequence* (z_n^i) *converges to* z^i *in* $\mathbb{X}_i$. *Let* W *be a neighborhood of* $z = (z^1, z^2, \ldots, z^k)$ *in* $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$. *According to proposition (3.30),* W *contains a neighborhood of the form* $N_1 \times \cdots \times N_k$, *where* N_i *is a neighborhood of* z^i *in* $\mathbb{X}_i$ *for all* $i = 1, \ldots, k$. *Thus, for all* $i = 1, \ldots, k$, *and for all* $N_i \in \mathcal{N}(z^i)$, *there exists* n_0^i *such that:*

$$n \geqslant n_0^i \Longrightarrow z_n^i \in N_i.$$

If we set $n_0 = \max(n_0^1, \ldots, n_0^k)$, *we obtain:*

$$n \geqslant n_0 \Longrightarrow z_n \in N_1 \times \cdots \times N_k,$$

which leads to:

$$n \geqslant n_0 \Longrightarrow z_n \in W.$$

Therefore, z_n *is a sequence converging to* z *in* $\mathbb{X}$.

Proposition 3.38. *If* $x = (x^1, \ldots, x^k)$ *is a cluster point of* (z_n) *in* $\mathbb{X} = \prod_{i=1}^{k} \mathbb{X}_i$, *then* x^i *is a cluster point of* (z_n^i) *for all* $i = 1, \ldots, k$.

Proof. *Let* $N_i \in \mathcal{N}(x^i)$ *for all* $i = 1, \ldots, k$, *then* $W = \mathbb{X}_1 \times \cdots \times \mathbb{X}_{i-1} \times N_i \times \mathbb{X}_{i+1} \times \cdots \times \mathbb{X}_k$ *is a neighborhood of* x *in* $\mathbb{X}$. *Consequently, we obtain:*

$$card\{n \in \mathbb{N} : z_n \in W\} = +\infty,$$

which leads to:

$$card\left\{n \in \mathbb{N} : z_n^i \in N_i\right\} = +\infty,$$

from which it follows that x^i *is a cluster point of* (z_n^i) *for all* $i = 1, \ldots, k$.

The previous result is generally false. For example, in $\mathbb{R}^2$, if we take the sequence $z_n = (x_n, y_n)$ defined by the following relations:

$$\begin{cases} x_{2n} & = n \\ x_{2n+1} & = \frac{1}{n} \end{cases}, \quad \begin{cases} y_{2n} & = \frac{1}{n} \\ y_{2n+1} & = n \end{cases}$$

It is clear that 0 is a cluster point of (x_n) and (y_n), but $(0,0)$ is not a cluster point of (z_n).

3.9 Continuous applications

Definition 3.27 (Pointwise continuity). *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. We say that a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is continuous at $x_0 \in \mathbb{X}$ if and only if for every neighborhood $N \in \mathcal{N}_{\mathbb{Y}}(f(x_0))$, there exists $U \in \mathcal{N}_{\mathbb{X}}(x_0)$ such that $f(U) \subseteq N$. In other words,*

$$\forall N \in \mathcal{N}_{\mathbb{Y}}(f(x_0)), \exists U \in \mathcal{N}_{\mathbb{X}}(x_0),\ f(U) \subseteq N \Longleftrightarrow f \text{ is continuous at } x_0. \tag{3.14}$$

Using the preimage, we obtain $U \subseteq f^{-1}(N)$, hence $f^{-1}(N)$ is a neighborhood of x_0. Therefore, we can write the previous definition in the following form.

Definition 3.28. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. We say that a function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ is continuous at $x_0 \in \mathbb{X}$ if and only if the preimage of any neighborhood of $f(x_0)$ in $\mathbb{Y}$ is a neighborhood of x_0 in $\mathbb{X}$. In other words,*

$$\forall N \in \mathcal{N}_{\mathbb{Y}}(f(x_0)),\ f^{-1}(N) \in \mathcal{N}_{\mathbb{X}}(x_0) \tag{3.15}$$

Remark 3.18. *In both previous definitions, we can replace $\mathcal{N}_{\mathbb{X}}(x_0)$ and $\mathcal{N}_{\mathbb{Y}}(f(x_0))$ with the basic neighborhoods of x_0 and $f(x_0)$.*

Example 3.33.

1. *The function $f : (\mathbb{R}, |.|) \longrightarrow (\mathbb{R}, \mathcal{P}(\mathbb{R}))$ such that for all $x \in \mathbb{R}$, $f(x) = x$, is not continuous on $\mathbb{R}$, because $N = \{x\}$ is a neighborhood of x in $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$, but $f^{-1}(N) = \{x\}$ is not a neighborhood of x in $(\mathbb{R}, |.|)$.*

2. *Let $\mathbb{X} = \{x_1, x_2, x_3, x_4\}$ and $\mathcal{T}_{\mathbb{X}} = \{\emptyset, \mathbb{X}, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3, x_4\}\}$, and let $\mathbb{Y} = \{y_1, y_2, y_3, y_4\}$ and $\mathcal{T}_{\mathbb{Y}} = \{\emptyset, \mathbb{Y}, \{y_1\}, \{y_1, y_2\}, \{y_1, y_2, y_3\}\}$. We define the function $f : \mathbb{X} \longrightarrow \mathbb{Y}$ by $f(x_4) = y_4$, $f(x_3) = y_2$, and $f(x_1) = f(x_2) = y_1$.*
   - *For example, we have $\mathcal{N}_{\mathbb{Y}}(f(x_4)) = \mathcal{N}_{\mathbb{Y}}(y_4) = \{\mathbb{Y}\}$, and $f^{-1}(\mathbb{Y}) = \mathbb{X} \in \mathcal{N}_{\mathbb{X}}(x_4)$, so f is continuous at x_4.*
   - *We also have $\mathcal{N}_{\mathbb{Y}}(y_2) = \{\{y_1, y_2\}, \{y_1, y_2, y_3\}, \{y_1, y_2, y_4\}, \mathbb{Y}\}$. If we take $N = \{y_1, y_2\}$, we obtain $f^{-1}(N) = \{x_1, x_2, x_3\} \notin \mathcal{N}_{\mathbb{X}}(x_3)$, so f is not continuous at x_3.*

Proposition 3.39 (Transitivity of continuity). *Let $\mathbb{X}, \mathbb{Y}$ and $\mathbb{T}$ be three topological spaces. Consider the two functions $f : \mathbb{X} \longrightarrow \mathbb{Y}$ and $g : \mathbb{Y} \longrightarrow \mathbb{T}$. If f is continuous at a point $x_0 \in \mathbb{X}$ and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.*

Proof. *Let $W \in \mathcal{N}_{\mathbb{T}}(g \circ f(x_0))$. Since g is continuous at $f(x_0)$, there exists $N \in \mathcal{N}_{\mathbb{Y}}(f(x_0))$ such that $g(N) \subseteq W$, and since f is continuous at x_0, there exists $U \in \mathcal{N}_{\mathbb{X}}(x_0)$ such that $f(U) \subseteq N$. From this, we deduce that $g \circ f(U) \subseteq W$, which implies that $g \circ f$ is continuous at x_0.*

Remark **3.19.** *The converse in the previous proposition is not always true.*
Consider the function f as shown in example (3.33(2)) and let $g : (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}}) \longrightarrow (\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ be a function defined as follows: $g(y_4) = x_4$, $g(y_3) = x_1$, $g(y_2) = x_3$, $g(y_1) = x_2$. On one hand, we have $\mathcal{N}_{\mathbb{X}}(g(f(x_4))) = \mathcal{N}_{\mathbb{X}}(g(y_4)) = \mathcal{N}_{\mathbb{X}}(x_4) = \{\{x_2, x_3, x_4\}, \mathbb{X}\}$. But, $g^{-1}(\{x_2, x_3, x_4\}) = \{y_1, y_2, y_4\} \notin \mathcal{N}_{\mathbb{Y}}(y_4)$, which means that g is not continuous at $f(x_4) = y_4$. On the other hand, we have $(g \circ f)(x_4) = g(f(x_4)) = g(y_4) = x_4$ and $\mathcal{N}_{\mathbb{X}}(x_4) = \{\{x_2, x_3, x_4\}, \mathbb{X}\}$, and $(g \circ f)^{-1}(\{x_2, x_3, x_4\}) = f^{-1}(g^{-1}(\{x_2, x_3, x_4\})) = f^{-1}(\{y_1, y_2, y_4\}) = \mathbb{X} \in \mathcal{N}_{\mathbb{X}}(x_4)$. Since $(g \circ f)^{-1}(\mathbb{X}) = \mathbb{X}$, we conclude that $g \circ f$ is continuous at x_4.

Proposition 3.40. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces and $f : \mathbb{X} \longrightarrow \mathbb{Y}$. The following statements are equivalent.*

1. *f is continuous.*
2. *$f(Cl(A)) \subseteq Cl(f(A))$ for every subset A of $\mathbb{X}$.*
3. *$f^{-1}(F)$ is closed in $\mathbb{X}$ for every closed set F in $\mathbb{Y}$.*
4. *$f^{-1}(O)$ is open in $\mathbb{X}$ for every open set O in $\mathbb{Y}$.*
5. *$f^{-1}(\beta)$ is open in $\mathbb{X}$ for every element β of a basis $\mathfrak{B}$ of $\mathcal{T}_{\mathbb{Y}}$.*
6. *$f^{-1}(IntB) \subseteq Intf^{-1}(B)$ for every subset B of $\mathbb{Y}$.*
7. *$Cl(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ for every subset B of $\mathbb{Y}$.*

Proof.

- ***(1)$\Longrightarrow$ (2)*** *Let $a \in Cl(A)$ and $N \in \mathcal{N}_{\mathbb{Y}}(f(a))$. Then $f^{-1}(N) \in \mathcal{N}_{\mathbb{X}}(a)$ because f is continuous. Consequently, $f^{-1}(N) \cap A \neq \emptyset$. Thus, if $x \in f^{-1}(N) \cap A$, we obtain $f(x) \in N \cap f(A)$, i.e., $N \cap f(A) \neq \emptyset$. Therefore, $f(a) \in Cl(f(A))$, which shows that $f(Cl(A)) \subset Cl(f(A))$.*
- ***(2)$\Longrightarrow$ (3)*** *Let F be a closed subset of $\mathbb{Y}$. Define $A = f^{-1}(F)$, so it is sufficient to show*

*that $A = Cl(A)$. By definition, we have $A \subseteq Cl(A)$, and according to **(2)**, we have $f(Cl(A)) \subseteq Cl(f(A)) \subseteq Cl(F) = F$ (since F is closed), hence $Cl(A) \subseteq f^{-1}(F) = A$. Consequently, $A = Cl(A)$, which shows that $f^{-1}(F)$ is closed in $\mathbb{X}$.*

- ***(3)$\Longrightarrow$ (4)** Let O be an open subset of $\mathbb{Y}$, then $\complement_{\mathbb{Y}} O$ is a closed set in $\mathbb{Y}$. Therefore, by **(3)**, the set $f^{-1}(\complement_{\mathbb{Y}} O)$ is closed in $\mathbb{X}$. Since $f^{-1}(\complement_{\mathbb{Y}} O) = \complement_{\mathbb{X}} f^{-1}(O)$, we deduce that $f^{-1}(O)$ is open in $\mathbb{X}$.*
- ***(4)$\Longrightarrow$ (5)** Obvious.*
- ***(5)$\Longrightarrow$ (6)** Let B be a subset of $\mathbb{Y}$. Then, $Int(B) = \bigcup_{i\in I} \beta_i$ such that $\{\beta_i : i \in I\}$ is a family of elements from a basis $\mathfrak{B}$ of $\mathcal{T}_{\mathbb{Y}}$. Using the inverse image, we obtain*

$$f^{-1}(Int(B)) = f^{-1}\left(\bigcup_{i\in I} \beta_i\right) = \bigcup_{i\in I} f^{-1}(\beta_i).$$

*Thus, $f^{-1}(Int(B))$ is an open set in $\mathbb{X}$ (according to **(5)**), and since $f^{-1}(Int(B)) \subseteq f^{-1}(B)$, we conclude that $f^{-1}(Int(B)) \subseteq Int(f^{-1}(B))$ (see Proposition (3.12)).*

- ***(6)$\Longrightarrow$ (7)**. Let B be a subset of $\mathbb{Y}$. Using Proposition (3.15(4)) and (6), we obtain:*

$$\complement_{\mathbb{X}} f^{-1}(Cl(B)) = f^{-1}(\complement_{\mathbb{Y}} Cl(B)) = f^{-1}(\mathrm{Int}\,\complement_{\mathbb{Y}} B)$$

$$\subseteq Int f^{-1}(\complement_{\mathbb{Y}} B) = Int \complement_{\mathbb{X}} f^{-1}(B) = \complement_{\mathbb{X}} Cl f^{-1}(B),$$

which shows that

$$Cl f^{-1}(B) \subseteq f^{-1}(Cl(B)).$$

- ***(7)$\Longrightarrow$ (1)** Let $x_0 \in \mathbb{X}$ and O be an open neighborhood of $f(x_0)$. Then, $\complement_{\mathbb{Y}} O$ is closed in $\mathbb{Y}$. Using **(7)**, we obtain*

$$Cl f^{-1}(\complement_{\mathbb{Y}} O) \subseteq f^{-1}(Cl(\complement_{\mathbb{Y}} O)) = f^{-1}(\complement_{\mathbb{Y}} O)$$

(since $\complement_{\mathbb{Y}} O$ is closed), and thus $f^{-1}(\complement_{\mathbb{Y}} O) = \complement_{\mathbb{X}} f^{-1}(O)$ is closed. Consequently, $f^{-1}(O)$ is open in $\mathbb{X}$. Finally, since $x_0 \in f^{-1}(O)$, we conclude that $f^{-1}(O) \in \mathcal{N}_{\mathbb{X}}(x_0)$, which shows that f is continuous.

Proposition 3.41. *Let $(A, \mathcal{T}_A)$ be a subspace of a topological space $(\mathbb{X}, \mathcal{T})$. Then the canonical injection $i : A \longrightarrow \mathbb{X}$ defined by $i(a) = a$, for all $a \in A$ is continuous.*

Proof. *Let O be an open set in $\mathbb{X}$. Then $i^{-1}(O) = O \cap A$, which is open in $(A, \mathcal{T}_A)$, so i is continuous.*

Proposition 3.42. *Let* $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be a continuous mapping and* $A \subset \mathbb{X}$. *Then the restriction* $f_{|A} : (A, \mathcal{T}_A) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *is continuous.*

Proof. *Given that* $f_{|A} = f \circ i$, *it follows that* $f_{|A}$ *is continuous because it is the composition of two continuous functions.*

Proposition 3.43. *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *and* $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be two topological spaces. If* $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *is continuous and injective, and* $\mathbb{Y}$ *is separated, then* $\mathbb{X}$ *is separated.*

Proof. *Let* $x, y \in \mathbb{X}$ *such that* $x \neq y$, *then* $f(x) \neq f(y)$ *(since* f *is injective), and since* $\mathbb{Y}$ *is separated, there exist two disjoint open sets* O_1 *and* O_2 *such that* $f(x) \in O_1$ *and* $f(y) \in O_2$. *Therefore,* $f^{-1}(O_1)$ *and* $f^{-1}(O_2)$ *are two disjoint open sets such that* $x \in f^{-1}(O_1)$ *and* $y \in f^{-1}(O_2)$, *which shows that* $\mathbb{X}$ *is separated.*

Definition 3.29 (Sequential Continuity). *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *and* $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be two topological spaces. We say that* f *is sequentially continuous at* x_0 *if for every sequence* (x_n) *that converges to* x_0, *the sequence* $f(x_n)$ *converges to* $f(x_0)$.

Remark 3.20. *We say that* f *is continuous (resp. sequentially continuous) on* $\mathbb{X}$ *if it is continuous (resp. sequentially continuous) at every point of* $\mathbb{X}$.

Proposition 3.44. *Every continuous function is sequentially continuous.*

Proof. *Let* f *be a function continuous at* x_0 *and let* (x_n) *be a sequence converging to* x_0. *Then, if* N *is a neighborhood of* $f(x_0)$, $f^{-1}(N)$ *is a neighborhood of* x_0, *so there exists* $n_0 \in \mathbb{N}$ *such that:*

$$n \geqslant n_0 \Rightarrow x_n \in f^{-1}(N),$$

or, equivalently,

$$n \geqslant n_0 \Rightarrow f(x_n) \in N,$$

which demonstrates that $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$.

Remark 3.21. *The converse in the previous proposition is not true in general.*

3.10 Open and closed maps

Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous function.
• If O is an open set in $\mathbb{X}$, then $f(O)$ is not necessarily open in $\mathbb{Y}$.
• If F is a closed set in $\mathbb{X}$, then $f(F)$ is not necessarily closed in $\mathbb{Y}$.
In other words, the continuous image of an open set (resp. closed set) is not necessarily an open set (resp. closed set).

Example 3.34.

1. *The function* $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ *defined by* $f(x) = \sin(x)$ *is continuous on* $\mathbb{R}$*, but* $f(\mathbb{R}) = [-1, 1]$ *is not an open set in* $\mathbb{R}$*.*

2. *The function* $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ *defined by* $f(x) = e^x$ *is continuous on* $\mathbb{R}$*, but* $f(\mathbb{R}) = (0, +\infty)$ *is not a closed set in* $\mathbb{R}$*.*

Definition 3.30. *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *and* $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be two topological spaces, and let* $f : \mathbb{X} \to \mathbb{Y}$*. We say that* f *is an open map (resp. closed map) if the image of every open set (resp. closed set) in* $\mathbb{X}$ *is an open set (resp. closed set) in* $\mathbb{Y}$*.*

Example 3.35.

1. *Let* $\mathbb{X}$ *be a topological space and* $A \subseteq \mathbb{X}$*. The canonical map* $i : (A, \mathcal{T}_A) \to \mathbb{X}$ *defined by* $i(x) = x$ *is open (resp. closed) if* A *is an open (resp. closed) subset of* $\mathbb{X}$*.*

2. *Let* $f : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ *be the function defined by* $f(x) = c \in \mathbb{R}$*. If* F *is closed in* $\mathbb{R}$*, then* $f(F) = \{c\}$ *is also closed in* $\mathbb{R}$*. However, if* O *is open in* $\mathbb{R}$*, then* $f(O) = \{c\}$ *is not open in* $\mathbb{R}$*. Therefore,* $f(x) = c$ *is a closed map but is not an open map.*

Proposition 3.45. *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *and* $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be two topological spaces, and let* $f : \mathbb{X} \to \mathbb{Y}$*. Then, for any* $A \subseteq \mathbb{X}$*, we have:*

1. f *is open* $\Longleftrightarrow f(IntA) \subseteq Int(f(A))$*.*

2. f *is closed* $\Longleftrightarrow Cl(f(A)) \subseteq f(ClA)$*.*

Proof.

1. $\Longrightarrow$) *Suppose that f is open; then $f(Int(A))$ is open in $\mathbb{Y}$. Consequently, $f(IntA) \subset Int(f(A))$ (since $Int(A) \subset A$).*
   $\Longleftarrow$) *Suppose that $f(IntA) \subset Int(f(A))$ and let A be an open set in $\mathbb{X}$. Then $f(A) = f(IntA) \subset Int(f(A))$, so $f(A) = Int(f(A))$, which shows that f is open.*

2. *Exercise: using arguments similar to those used in **(1)**.*

Proposition 3.46. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$. Then, for any $A \subseteq \mathbb{X}$ and $B \subseteq \mathbb{Y}$, we have:*

1. *f is continuous and open $\Longleftrightarrow f^{-1}(IntB) = Int(f^{-1}(B))$.*

2. *f is continuous and closed $\Longleftrightarrow Cl(f(A)) = f(ClA)$.*

Proof.

1. $\Longrightarrow$) *Suppose f is open and continuous. Then we obtain*

   (i) $$f^{-1}(IntB) \subset Int(f^{-1}(B)),$$

   according to Proposition (3.40(6)). On the other hand, since $Int(f^{-1}(B))$ is open in $\mathbb{X}$, we have that $f(Int(f^{-1}(B)))$ is open in $\mathbb{Y}$ (since f is open). Consequently, $f(Int(f^{-1}(B))) = Intf(Int(f^{-1}(B))) \subseteq Int(f(f^{-1}(B))) \subseteq IntB$, so

   (ii) $$Int(f^{-1}(B)) \subseteq f^{-1}(IntB).$$

   Finally, the two inclusions (i) and (ii) show that $f^{-1}(IntB) = Int(f^{-1}(B))$.

   $\Longleftarrow$) *Suppose $f^{-1}(IntB) = Int(f^{-1}(B))$. Then f is continuous (see Proposition 3.40(6)). Moreover, if A is an open set in $\mathbb{X}$, we have*

   $$A = Int(A) \subset Int(f^{-1}(f(A))) = f^{-1}(Int(f(A)),$$

   and thus $f(A) \subset Int(f(A))$. Hence, $f(A)$ is open, so f is open.

2. *Clear (using Proposition (3.40(2)) and Proposition (3.45(**2**))).*

3.11 Homeomorphism

Definition 3.31. *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *and* $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ *be two topological spaces and* $f : \mathbb{X} \longrightarrow \mathbb{Y}$*. We say that* f *is an homeomorphism from* $\mathbb{X}$ *to* $\mathbb{Y}$ *if:*

1. *f is a bijection (one-to-one and onto),*
2. *f is continuous,*
3. *the inverse function f^{-1} is continuous (f is an open mapping).*

If there exists an homeomorphism from $\mathbb{X}$ *to* $\mathbb{Y}$*, we say that* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic or topologically equivalent, and we denote this by* $\mathbb{X} \cong \mathbb{Y}$*. Any property preserved by an homeomorphism is called a topological property.*

Example 3.36.

1. *Let* $\mathbb{X} = \mathbb{R}$ *and* $\mathbb{Y} = (-1, 1)$ *endowed with the usual topology. The function* $f : \mathbb{R} \longrightarrow (-1, 1)$ *defined by* $f(x) = \dfrac{x}{1 + |x|}$ *is a homeomorphism. Consequently,* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic.*
2. *Let* $\mathbb{X} = (a, b)$ *and* $Y = \mathbb{R}$ *with the usual topology. The function* $f : (a, b) \longrightarrow \mathbb{R}$ *defined by* $f(x) = \dfrac{1}{x - a} + \dfrac{1}{x - b}$ *is a homeomorphism. Therefore,* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic.*
3. *Let* $\mathbb{X} = (0, 1)$ *and* $\mathbb{Y} = (a, b)$ *endowed with the usual topology. The function* $f : (0, 1) \longrightarrow (a, b)$ *defined by* $f(x) = (b - a)x + a$ *is a homeomorphism. Consequently,* $\mathbb{X}$ *and* $\mathbb{Y}$ *are homeomorphic.*

Remark 3.22.

1. *In general, the bijectivity and continuity of f do not imply that f is a homeomorphism. For example, the map* $f : (\mathbb{R}, \mathcal{P}(\mathbb{R})) \longrightarrow (\mathbb{R}, |\cdot|)$ *defined by* $f(x) = x$ *is a bijection and continuous, while* f^{-1} *is not continuous.*
2. *Homeomorphisms are, by definition, open and closed maps.*

BIBLIOGRAPHY

[1] N. Bourbaki. *Topologie générale, Chapitres 1 à 4.* Hermann, Paris, 1971.

[2] H. Brezis. *Analyse Fonctionnelle. Théorie et applications.* Masson„ (1983).

[3] G. Choquet. *Cours d'analyse, tome II, Topologie.* Masson, Paris, 1964.

[4] G. Christol. *Topologie.* Ellipses, 1997.

[5] J. Dieudonné. *Éléments d'analyse, tome I : fondements de l'analyse moderne.* Gauthier-Villars, Paris, 1968.

[6] J. Dixmier. *Topologie générale.* Presses universitaires de France, 1981.

[7] J.L. Kelley. *General Topology.* Van Nostrand, (1955).

[8] F. Nier, D. Iftimie. *Introduction à la Topologie.* Université de Rennes 1.

[9] C. Tisseron. *Notions de Topologie. Introduction aux espaces fonctionnels.* Hermann (1985).

[10] B. Yallaoui. *Topologie Générale Elémentaire.* Université Ferhat Abbas, Sétif 1.