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INTRODUCTION TO METRIC & TOPOLOGICAL SPACES

FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS

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CONTENTS

CHAPTER 4

COMPACT SPACES

4.1 Compactness in Topological Spaces

4.1.1 Compact Spaces and Sets

Let $(\mathbb{X}, \mathcal{T})$ be a topological space and $\{O_i : i \in I\}$ a family of open sets in $\mathbb{X}$.

Definition 4.1. *We say that the family* $\{O_i : i \in I\}$ *is an open cover of* $\mathbb{X}$ *if* $\mathbb{X} = \bigcup_{i \in I} O_i$.

Definition 4.2. *We say that the family* $\{O_i : i \in I\}$ *is an open cover of a subset* A *of* $\mathbb{X}$ *if* $A \subseteq \bigcup_{i \in I} O_i$.

Definition 4.3 (**Borel-Lebesgue**). *The topological space* $(\mathbb{X}, \mathcal{T})$ *is said to be compact if it is Hausdorff (separated) and for every open cover* $\{O_i : i \in I\}$ *of* $\mathbb{X}$*, one can extract a finite subcover. In other words:*

$$\left(\mathbb{X} = \bigcup_{i \in I} O_i\right) \Longrightarrow \left(\exists J \ (\textit{finite}) \subset I \ \textit{such that} \ \mathbb{X} = \bigcup_{i \in J} O_i\right). \tag{4.1}$$

The following definition characterize compactness in terms of closed subsets of the space.

Definition 4.4. *The topological space* $(\mathbb{X}, \mathcal{T})$ *is said to be* compact *if it is Hausdorff (separated), and for every family of closed sets* $\{F_i : i \in I\}$ *in* $\mathbb{X}$ *with an empty intersection, one can extract a finite subfamily whose intersection is also empty. In other words:*

$$\left(\bigcap_{i\in I} F_i = \emptyset\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } \bigcap_{i\in J} F_i = \emptyset\right). \tag{4.2}$$

Example 4.1.

1. *The space* $(\mathbb{R}, |.|)$ *is Hausdorff, but it is not compact because the family* $\{(-n, +n) : n \in \mathbb{N}\}$ *is an open cover of* $\mathbb{R}$ *that does not have any finite subcover of* $\mathbb{R}$.
2. *The space* $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ *is Hausdorff, but it is not compact because the family* $\{\{x\} : x \in \mathbb{R}\}$ *is an open cover of* $\mathbb{R}$ *that does not have any finite subcover of* $\mathbb{R}$.
3. *Any finite Hausdorff space is compact.*

Definition 4.5. *A subset* A *of a Hausdorff topological space* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *is said to be compact if the subspace topology* $(A, \mathcal{T}_A)$ *is compact. In other words:*

$$\left(A \subset \bigcup_{i\in I} O_i\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } A \subset \bigcup_{i\in J} O_i\right). \tag{4.3}$$

Remark 4.1. *The* Borel-Lebesgue *property in* $(A, \mathcal{T}_A)$ *is expressed using the open sets of* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *in the form (4.3).*

Proposition 4.1. *A subset* A *of a Hausdorff topological space* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *is compact if and only if for every family of closed sets* $\{F_i : i \in I\}$ *in* $\mathbb{X}$*, we have:*

$$\left(A \cap \left(\bigcap_{i\in I} F_i\right) = \emptyset\right) \Longrightarrow \left(\exists J \text{ (finite) } \subset I \text{ such that } A \cap \left(\bigcap_{i\in J} F_i\right) = \emptyset\right). \tag{4.4}$$

Proof.

$\Longrightarrow$) *On the one hand, if A is compact then we have:*

$$\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right)\Longrightarrow\left(A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in I}F_i\right)=\bigcup_{i\in I}\complement_{\mathbb{X}}F_i\right)$$

Using definition ([4.5](#)), since $\left\{\complement_{\mathbb{X}}F_i : i\in I\right\}$ is a family of open sets in $\mathbb{X}$, we deduce that:

$$\exists J\ \textit{(finite)}\ \subset I\ \textit{such that}\ A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i.$$

This shows that:

$$\exists J\ \textit{(finite)}\ \subset I\ \textit{such that}\ A\cap\left(\bigcap_{i\in J}F_i\right)=\emptyset.$$

$\Longleftarrow$) *On the other hand, we have:*

$$\left(A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i\right)\Longleftrightarrow\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right).$$

Now, using ([4.4](#)) we obtain:

$$\left(A\cap\left(\bigcap_{i\in I}F_i\right)=\emptyset\right)\Longrightarrow\left(\exists J\ \textit{(finite)}\ \subset I\ \textit{such that}\ A\cap\left(\bigcap_{i\in J}F_i\right)=\emptyset\right)$$

$$\Longrightarrow\left(\exists J\ \textit{(finite)}\ \subset I\ \textit{such that}\ A\subset\complement_{\mathbb{X}}\left(\bigcap_{i\in J}F_i\right)=\bigcup_{i\in J}\complement_{\mathbb{X}}F_i\right).$$

Since $\left\{\complement_{\mathbb{X}}F_i : i\in I\right\}$ is a family of open sets in $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$, we conclude that A is compact.

Example 4.2.

1. *$A=(0,1]$ is not compact because $I_n=(\frac{1}{n},1]$ is a sequence of open sets in A covering A, and no finite subcover can be extracted.*

2. *Any finite subset of a Hausdorff space is compact.*

4.1.2 Properties of Compact Topological Spaces

Proposition 4.2. *In a Hausdorff topological space, a compact subset is closed.*

Proof. *Let K be a compact subset in a Hausdorff topological space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$. It suffices to show that $\complement_{\mathbb{X}} K$ is open. Let $x \in \complement_{\mathbb{X}} K$. Since $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ is Hausdorff, for every $y \in K$, there exist two open sets $N_{x,y} \in \mathcal{N}(x)$ and $W_{x,y} \in \mathcal{N}(y)$ such that $N_{x,y} \cap W_{x,y} = \emptyset$. The family $\{W_{x,y} : y \in K\}$ is an open cover of K and is compact, so we can extract a finite subcover $\{W_{x,y_i} : i = 1, \dots, n\}$ such that $K \subset \bigcup_{i=1}^{n} W_{x,y_i}$. If we take $N = \bigcap_{i=1}^{n} N_{x,y_i}$, we obtain $N \in \mathcal{N}(x)$ and $N \subset \complement_{\mathbb{X}} K$, which shows that $\complement_{\mathbb{X}} K$ is open (because it is a neighborhood of each of its points).*

Proposition 4.3. *If $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ is a compact topological space and $F \subset \mathbb{X}$, then F is compact if and only if F is a closed subset of $\mathbb{X}$.*

Proof.

$\Longrightarrow$ *) This is evident from the previous proposition.*
$\Longleftarrow$ *) Suppose that F is a closed subset of $\mathbb{X}$. Then, if $\{F_i : i \in I\}$ is a family of closed subsets of X such that $F \cap \left(\bigcap_{i \in I} F_i \right) = \emptyset$, we obtain $\bigcap_{i \in I} (F \cap F_i) = \emptyset$. Therefore, by definition (4.4), there exists a finite subset $J \subset I$ such that $\emptyset = \bigcap_{i \in J} (F \cap F_i) = F \cap \left(\bigcap_{i \in J} F_i \right)$. Thus, F is compact by proposition (4.1).*

Proposition 4.4. *In a Hausdorff topological space, a finite union of compact sets is compact.*

Proof. *Let $\{K_k : k = 1, \dots, n\}$ be a finite family of compact sets in a topological space $\mathbb{X}$ and let $K = \bigcup_{k=1}^{n} K_k$. Then, any open cover $\{O_i : i \in I\}$ of K is also an open cover of each K_k, for each $k = 1, \dots, n$. Therefore, there exists a finite subset $J_k \subset I$ such that $K_k \subset \bigcup_{i \in J_k} O_i$, for each $k = 1, \dots, n$. Taking $J = J_1 \cup \dots \cup J_n$, we see that $\bigcup_{i \in J} O_i$ is a finite subcover of K, which shows that K is compact.*

Proposition 4.5. *In a Hausdorff topological space, any intersection of compact sets is compact.*

Proof. *Let* $\{K_i : i \in I\}$ *be a family of compact sets in a Hausdorff topological space* $\mathbb{X}$*, and let* $K = \bigcap_{i\in I} K_i$*. Then* K *is closed (since it is an intersection of closed sets) within the compact set* K_{i_0} *for some* $i_0 \in I$*. Therefore, by proposition (4.3),* K *is compact.*

Lemma 4.1 (Bolzano-Weierstrass). *Let* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *be a compact topological space. Then, every infinite subset of* $\mathbb{X}$ *has at least one accumulation point.*

Proof. *If* A *is an infinite subset of* $\mathbb{X}$ *with no accumulation points, then for each* $x \in \mathbb{X}$*, there exists an open neighborhood* $N_x \in \mathcal{N}(x)$ *such that*

$$N_x \cap A = \begin{cases} \{x\} & \text{if } x \in A, \\ \emptyset & \text{if } x \notin A. \end{cases}$$

Thus, the family $\{N_x : x \in \mathbb{X}\}$ *forms an open cover of* $\mathbb{X}$*, which is compact. Therefore, we can extract a finite subcover* $\{N_{x_i} : i = 1, \dots, n\}$ *such that*

$$\mathbb{X} = \bigcup_{i=1}^{n} N_{x_i}.$$

However, we also have

$$A = A \cap \mathbb{X} = A \cap \left(\bigcup_{i=1}^{n} N_{x_i} \right) = \bigcup_{i=1}^{n} (A \cap N_{x_i}),$$

which implies that A *contains at most* n *elements, contradicting the assumption that* A *is infinite.*

Lemma 4.2 (Weierstrass). *Let* $(\mathbb{X}, \mathcal{T}_X)$ *and* $(\mathbb{Y}, \mathcal{T}_Y)$ *be two topological spaces such that* $(Y, \mathcal{T}_Y)$ *is Hausdorff, and let* $f : \mathbb{X} \to \mathbb{Y}$ *be a continuous map. If* A *is a compact subset of* $\mathbb{X}$*, then* $f(A)$ *is a compact subset of* $\mathbb{Y}$*.*

Proof. *Let* $\{O_i : i \in I\}$ *be an open cover of* $f(A)$*, i.e.,* $f(A) \subset \bigcup_{i\in I} O_i$*. Since* f *is continuous, the family* $\{f^{-1}(O_i) : i \in I\}$ *is an open cover of* A*. By the compactness of* A*, there exists a finite subset* $J \subset I$ *such that*

$$A \subset \bigcup_{i\in J} f^{-1}(O_i) = f^{-1}\left(\bigcup_{i\in J} O_i \right).$$

Since $f(A) \subset f\left(f^{-1}\left(\bigcup\limits_{i\in J} O_i\right)\right) \subset \bigcup\limits_{i\in J} O_i$, *we conclude that* $f(A)$ *is a compact subset of* $\mathbb{Y}$.

Remark 4.2. *According to the previous proposition, we conclude that compactness is a topological property.*

The following corollary is a version of the extreme value theorem.

Corollaire 4.1 (Heine). *If* $(\mathbb{X}, \mathcal{T})$ *is a compact topological space and* $f : \mathbb{X} \to (\mathbb{R}, |.|)$ *is a continuous function, then* f *is bounded on* $\mathbb{X}$, *and there exist points* $a, b \in \mathbb{X}$ *such that*

$$f(a) = \max_{x\in\mathbb{X}} f(x) \quad \textit{and} \quad f(b) = \min_{x\in\mathbb{X}} f(x).$$

Proof. *Since* f *is continuous and* $\mathbb{X}$ *is compact,* $f(\mathbb{X})$ *is a compact subset of* $(\mathbb{R}, |.|)$ *(see Lemma (4.2)). It follows that* $f(X)$ *is closed and bounded (see Proposition (4.2)). Let* $M = \sup f(\mathbb{X})$. *Since* $f(\mathbb{X})$ *is closed, we conclude that* $M \in f(\mathbb{X})$, *and therefore, there exists* $a \in \mathbb{X}$ *such that* $f(a) = M = \max\limits_{x\in\mathbb{X}} f(x)$. *Similarly, we can show that the minimum is attained.*

Remark 4.3. *The previous corollary shows that continuous functions on a compact set and with values in* $\mathbb{R}$ *attain their bounds.*

Proposition 4.6. *Let* $(\mathbb{X}, \mathcal{T}_\mathbb{X})$ *be a compact space,* $(\mathbb{Y}, \mathcal{T}_\mathbb{Y})$ *be a Hausdorff space, and* $f : (\mathbb{X}, \mathcal{T}_\mathbb{X}) \longrightarrow (\mathbb{Y}, \mathcal{T}_\mathbb{Y})$ *be a continuous function, then* f *is closed.*

Proof. *Let* F *be a closed subset of* $\mathbb{X}$, *then* F *is compact (see proposition (4.3)), and thus* $f(F)$ *is compact (see lemma (4.2)), from which it follows that* $f(F)$ *is closed (see proposition (4.2)).*

Proposition 4.7. *Let* $(\mathbb{X}, \mathcal{T}_\mathbb{X})$ *be a compact space,* $(\mathbb{Y}, \mathcal{T}_\mathbb{Y})$ *be a Hausdorff space, and* $f : (\mathbb{X}, \mathcal{T}_\mathbb{X}) \longrightarrow (\mathbb{Y}, \mathcal{T}_\mathbb{Y})$ *be a continuous bijection, then* f *is a homeomorphism.*

Proof. *It is enough to show that* f^{-1} *is continuous. Let* $g = f^{-1}$. *If* F *is a closed set in* $\mathbb{X}$, *then it is compact, and thus* $f(F)$ *is compact. However, a compact subset of a separated space is closed, so* $g^{-1}(F) = f(F)$ *is closed. Therefore,* $g = f^{-1}$ *is continuous.*

Theorem 4.1 (Tychonoff's Theorem). *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces, then $\prod_{i\in I} \mathbb{X}_i$ is compact if and only if $\mathbb{X}_i$ is compact for all $i \in I$.*

Proof. *We will assume the result in the general case. Here, we simply prove it for the finite product of compact spaces. Therefore, it is sufficient to prove it for the product of two compact spaces.*

$\Longrightarrow$) *If $\mathbb{X} \times \mathbb{Y}$ is compact, then $\mathbb{X}$ and $\mathbb{Y}$ are compact because they are the images of the continuous projections $P_{\mathbb{X}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{X}$ and $P_{\mathbb{Y}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{Y}$.*

$\Longleftarrow$) *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are compact. Let $\{O_i : i \in I\}$ be an open cover of $\mathbb{X} \times \mathbb{Y}$. Then, for every $(x,y) \in \mathbb{X} \times \mathbb{Y}$, there exist $U_{(x,y)} \in \mathcal{T}_{\mathbb{X}}$ and $V_{(x,y)} \in \mathcal{T}_{\mathbb{Y}}$ such that $(x,y) \in U_{(x,y)} \times V_{(x,y)} \subseteq O_{(x,y)}$ with $O_{(x,y)} \in \{O_i : i \in I\}$.*

Notice that for each $x \in \mathbb{X}$, the family $\left\{V_{(x,y)} : y \in \mathbb{Y}\right\}$ is an open cover of the compact space $\mathbb{Y}$, and so we can extract a finite subcover $\left\{V_{(x,y_i)} : i = 1, \ldots, n\right\}$ for it. On the other hand, if we take $W_x = \bigcap_{i=1}^{n} U_{(x,y_i)}$, then the family $\{W_x : x \in \mathbb{X}\}$ is an open cover of the compact space $\mathbb{X}$, and so we can extract a finite subcover $\left\{W_{x_j} : j = 1, \ldots, m\right\}$.

We deduce that the family $\left\{W_{x_j} \times V_{(x_j,y_i)} : i = 1, \ldots, n, j = 1, \ldots, m\right\}$ is a finite cover of $\mathbb{X} \times \mathbb{Y}$. Moreover, we have:

$$W_{x_j} \times V_{(x_j,y_i)} \subseteq U_{(x_j,y_i)} \times V_{(x_j,y_i)} \subseteq O_{(x_j,y_i)}, \quad \forall 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m.$$

Thus, the family $\left\{O_{(x_j,y_i)} : i = 1, \ldots, n \text{ and } j = 1, \ldots, m\right\}$ is a finite open cover of $\mathbb{X} \times \mathbb{Y}$, which shows that $\mathbb{X} \times \mathbb{Y}$ is compact.

Definition 4.6 (Relative compactness). *A set A is said to be relatively compact in a topological space $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ if $Cl(A)$ is compact.*

Example 4.3.

1. *Every non-empty subset of a compact space is relatively compact.*

2. *Every compact set is relatively compact.*

Definition 4.7 (Local compactness). *A space* $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ *is said to be locally compact if it is Hausdorff and every point of* $\mathbb{X}$ *has at least one compact neighborhood.*

Example 4.4.

1. $(\mathbb{R}, |\cdot|)$ *is locally compact because it is Hausdorff and* $[x-r, x+r]$ *is a compact neighborhood of every* $x \in \mathbb{R}$.

2. *Every discrete space is locally compact because it is Hausdorff and* $\{x\}$ *is a compact neighborhood of every point* x *in this space.*

4.2 Compactness in metric spaces

The definitions of compactness in a metric space $(\mathbb{X}, d)$ are the same as those we saw in a topological space (see the previous section (4.1)).

Remark 4.4. *In an abstract topological space, there is no notion of distance, and therefore we do not talk about bounded sets.*

4.2.1 Precompact spaces and sequentially compact spaces

Definition 4.8. *Let* A *be a subset of a metric space* $(\mathbb{X}, d)$. *We say that* A *is bounded if there exists a ball* $B(x,r)$ *with center* x *and radius* $r > 0$ *such that* $A \subseteq B(x,r)$.

Definition 4.9. *Let* $(\mathbb{X}, d)$ *be a metric space and* A *a subset of* $\mathbb{X}$. *We say that* A *is sequentially compact if every sequence in* A *has a convergent sub-sequence.*

Definition 4.10. *Let* $(\mathbb{X}, d)$ *be a metric space and* A *a subset of* $\mathbb{X}$. *We say that* A *is precompact (or totally bounded) if for every* $r > 0$, *there exist points* $x_1, \ldots, x_n$ *in* A *such that* $A \subseteq \bigcup_{i=1}^{n} B(x_i, r)$.

Remark 4.5. *According to the two definitions (4.8) and (4.10), every precompact set is bounded.*

Example 4.5.

1. *Any finite subset A of a metric space $(\mathbb{X}, d)$ is sequentially compact because if $(x_n)_{n\in\mathbb{N}}$ is a sequence in A, then at least one of the elements $x \in A$ must repeat infinitely many times in this sequence, and thus the sequence $(x_0, \ldots, x_i, x, x, \ldots)$ is convergent.*

2. *Every finite subset A of a metric space $(\mathbb{X}, d)$ is precompact, because for any $r > 0$, there exist points $x_1, \ldots, x_n$ in A such that:$A \subseteq \bigcup\limits_{i=1} B(x_i, r)$.*

3. *If $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to x_0 in a metric space $(\mathbb{X}, d)$, then the set $A = \{x_n : n \geqslant 0\} \cup \{x_0\}$ is precompact. This is because for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that*

$$\forall n \in \mathbb{N}, n \geqslant n_0 \Longrightarrow d(x_n, x_0) < \varepsilon.$$

Therefore,

$$A \subseteq \bigcup_{k=1}^{n_0-1} B(x_k, \varepsilon) \cup B(x_0, \varepsilon).$$

Lemma 4.3. *Let $(\mathbb{X}, d)$ be sequentially compact metric space. If $\{O_i : i \in I\}$ is an open cover of $\mathbb{X}$, then there exists $r > 0$ such that for every $x \in \mathbb{X}$, the ball $B(x, r)$ is contained in one of the open sets O_i.*

Proof. *Suppose that for every $n \in \mathbb{N}^*$, there exists a point $x_n \in \mathbb{X}$ such that the ball $B(x_n, \frac{1}{n})$ is not contained in any of the open sets O_i, i.e., $B(x_n, \frac{1}{n}) \cap \complement_{\mathbb{X}} O_i \neq \emptyset$ for all $i \in I$. Since $(\mathbb{X}, d)$ is sequentially compact, the sequence (x_n) has a convergent subsequence (x_{n_k}). Let $x_{n_k} \to x \in \mathbb{X}$. Then, there exists at least one $i_0 \in I$ such that $x \in O_{i_0}$, and therefore there exists $r > 0$ such that $B(x, r) \subset O_{i_0}$.*

On the other hand, since $x_{n_k} \to x$, the ball $B(x, \frac{r}{2})$ contains infinitely many points of (x_{n_k}). Thus, there exists $n_p > \frac{2}{r}$ such that $x_{n_p} \in B(x, \frac{r}{2})$, which implies that $B(x_{n_p}, \frac{1}{n_p}) \subset B(x, r) \subset O_{i_0}$, leading to a contradiction.

4.2.2 Properties of Compact Metric Spaces

Proposition 4.8. *Let $(\mathbb{X}, d)$ be a metric space and A a closed subset of $\mathbb{X}$. Then, the following statements are equivalent.*

1. *A is compact.*
2. *If $\mathcal{F} = \{F_i : i \in I\}$ is a family of closed subsets of A such that for every finite family sets $F_1, \ldots, F_n \in \mathcal{F}$ we have $\bigcap_{i=1}^{n} F_i \neq \emptyset$, then $\bigcap_{i \in I} F_i \neq \emptyset$.*
3. *A is sequentially compact.*
4. *Every infinite subset of A has an accumulation point.*
5. *A is complete and precompact.*

Proof.

Let us assume that A is compact and let $\mathcal{F} = \{F_i : i \in I\}$ be a family of closed subsets of A such that for every finite collection $F_1, \ldots, F_n \in \mathcal{F}$, we have $\bigcap_{i=1}^{n} F_i \neq \emptyset$. Now, if $\bigcap_{i \in I} F_i = \emptyset$, it follows that the family $\{\complement_{\mathbb{X}} F_i : i \in I\}$ is an open cover of $\mathbb{X}$ and, therefore, of A. Since A is compact, there exist $F_1, \ldots, F_n \in \mathcal{F}$ such that $A \subseteq \bigcup_{i=1}^{n} \complement_{\mathbb{X}} F_i = \complement_{\mathbb{X}} \bigcap_{i=1}^{n} F_i$. Given that $F_i \subseteq A$ for every $i = 1, \ldots, n$, we conclude that $\bigcap_{i=1}^{n} F_i = \emptyset$, which contradicts the assumption that $\bigcap_{i=1}^{n} F_i \neq \emptyset$.
(2) $\Longrightarrow$ ***(1)*** *Let $\{O_i : i \in I\}$ be an open cover of A, i.e., $A \subseteq \bigcup_{i \in I} O_i$, and thus $\bigcap_{i \in I} \complement_A O_i = \emptyset$, which implies $\bigcap_{i=1}^{n} \complement_A O_i = \emptyset$ (according to **(2)**), showing that $A \subseteq \bigcup_{i=1}^{n} O_i$, and hence A is a compact set.*
(3) $\Longrightarrow$ ***(4)*** *If K is an infinite subset of A, then K contains a sequence of distinct points (x_n); by **(3)**, there exists a subsequence (x_{n_k}) converging to x. Therefore, x is an accumulation point of K.*
(4) $\Longrightarrow$ ***(3)*** *Let (x_n) be a sequence of distinct points in A. Using equation (4), we conclude that (x_n) has an accumulation point $x \in A$ because A is closed. The ball $B(x,1)$ contains infinitely many elements of the sequence (x_n), so we choose $x_{n_1} \in B(x,1)$. The ball $B(x,\frac{1}{2})$ contains infinitely many elements of the sequence (x_n), so we choose $x_{n_2} \in B(x,\frac{1}{2})$ with $n_2 > n_1$. We repeat this process, choosing $x_{n_3} \in B(x,\frac{1}{3})$ with $n_3 > n_2$. Therefore, we can select a subsequence (x_{n_k}), such that $n_{k+1} > n_k$ and $x_{n_k} \in B(x,\frac{1}{k})$ for all $k = 1,2,\ldots$. It is clear that this subsequence converges to x.*
(1) $\Longrightarrow$ ***(4)*** *(See Lemma (4.1))*
(1) $\Longrightarrow$ ***(5)*** *Assume that A is compact.*

- *Let (x_n) be a Cauchy sequence in A. Since **(1)** $\Longrightarrow$ **(4)** $\Longleftrightarrow$ **(3)**, there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x$, so $x_n \to x$, and hence A is complete.*
- *For every $r > 0$, the family $\{B(x,r) : x \in A\}$ is an open cover of A, so there exists a finite open subcover $\{B(x_i,r) : i = 1,\ldots,n\}$ of A, that is, there exist points $x_1,\ldots,x_n$ in A such that $A \subseteq \bigcup_{i=1}^{n} B(x_i,r)$, which shows that A is precompact.*

(5) $\Longrightarrow$ ***(3)*** *Let (x_n) be a sequence in A and (r_n) a decreasing sequence of positive numbers such that $(r_n) \to 0$. Using **(5)**, we conclude that there exists a finite cover of A by the balls $\{B(x_i,r_1) : i = 1,\ldots,n\}$. Therefore, there exists a ball $B(y_1,r_1)$ that contains infinitely many elements of (x_n). Let $\mathbb{N}_1 = \{n \in \mathbb{N} : d(y_1,x_n) < r_1\}$. Now, consider the sequence*

$\{x_n : n \in \mathbb{N}_1\}$ and the balls of radius r_2. We repeat the process; there exists $y_2 \in A$ such that $\mathbb{N}_2 = \{n \in \mathbb{N}_1 : d(y_2, x_n) < r_2\}$ is an infinite set. By induction, we can show that for each $i \geq 1$, we choose a point $y_k \in A$ and an infinite set $\mathbb{N}_k$ such that $\mathbb{N}_{k+1} \subset \mathbb{N}_k$ and $\{x_n : n \in \mathbb{N}_k\} \subset B(y_k, r_k)$. If we define $F_k = Cl\{x_n : n \in \mathbb{N}_k\}$, then $F_{k+1} \subset F_k$ and $diam(F_k) \leqslant 2r_k$. Since A is complete, Cantor's theorem implies that $\bigcap_k F_k = \{x\}$. If we choose $n_k \in \mathbb{N}_k$, then (x_{n_k}) is a subsequence of (x_n) converging to x, and hence A is sequentially compact.

(5)* $\Longrightarrow$ *(1) *Let $G = \{O_i : i \in I\}$ be an open cover of A. Since A is precompact, for every $r > 0$, there exist points $x_1, \ldots, x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_k, r)$. But for each $1 \leq k \leq n$, there exists $O_k \in G$ such that $x_k \in O_k$. Thus, it is sufficient to choose $r > 0$ such that $B(x_k, r) \subset O_k$ for all $1 \leq k \leq n$ (see Lemma 4.3). We then deduce that $A \subseteq \bigcup_{i=1}^{n} B(x_k, r) \subset \bigcup_{i=1}^{n} O_k$, which shows that the family $\{O_k : k = 1, \ldots, n\}$ is a finite open cover of A, and therefore A is compact.*

(3)* $\Longrightarrow$ *(5) *Let (x_n) be a Cauchy sequence in A, then **(3)** implies that there exists a subsequence (x_{n_k}) such that $x_{n_k} \to x$. Since $d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x)$, it follows that $x_n \to x$.*

Proposition 4.9.

1. *For all $a, b \in \mathbb{R}$, the closed interval $[a, b]$ is compact in $(\mathbb{R}, |\cdot|)$.*
2. *A subset A of $(\mathbb{R}, |\cdot|)$ is compact if and only if it is closed and bounded.*

Proof.

1. *The interval $[a, b]$ is closed in $\mathbb{R}$, which is complete, and therefore complete. Thus, according to the previous proposition, it is enough to show that it is precompact. Indeed, for every $r > 0$, we can find points $a = x_1, x_2, \ldots, x_n = b$ such that $x_i - x_{i-1} < r$, and $[a, b] \subseteq \bigcup_{i=1}^{n} (x_i - r, x_i + r)$.*
2.

   $\Longrightarrow$ *) Let A be a compact subset of $(\mathbb{R}, |\cdot|)$, then it is complete and precompact, according to the previous theorem, and therefore it is closed and bounded (see Proposition* **??** *and Remark 4.5).*

   $\Longleftarrow$ *) Let A be a closed and bounded subset of $(\mathbb{R}, |\cdot|)$, then there exists a closed and bounded interval $[a, b]$ such that $A \subseteq [a, b]$, and since $[a, b]$ is compact, A is compact (see Proposition 4.3).*

Example 4.6.

1. *Every bounded and closed subset of $(\mathbb{R}^2, d_2)$ is compact.*
2. *Any subset of $(\mathbb{R}^2, d_2)$ that is either unbounded or not closed is not compact.*

3. *The closed disk* $\left\{x \in \mathbb{R}^2 : d_2(x,y) \leqslant r\right\}$ *is compact.*

Lemma 4.4 (Heine). *If* $f : (\mathbb{X}, d_{\mathbb{X}}) \to (\mathbb{Y}, d_{\mathbb{Y}})$ *is continuous and* $\mathbb{X}$ *is compact, then* f *is uniformly continuous.*

Proof. *Suppose that* f *is continuous but not uniformly continuous. Then, there exist* $\varepsilon > 0$ *and two sequences* (x_n) *and* (y_n) *in* $\mathbb{X}$ *such that:*

$$d_{\mathbb{X}}(x_n, y_n) < \frac{1}{n} \quad \text{and} \quad d_{\mathbb{Y}}(f(x_n), f(y_n)) \geqslant \varepsilon.$$

Since $\mathbb{X}$ *is compact, there exists* $x \in \mathbb{X}$ *and a subsequence* x_{n_k} *such that* $d(x_{n_k}, x) \longrightarrow 0$. *We deduce that*

$$d(y_{n_k}, x) \leqslant d(y_{n_k}, x_{n_k}) + d(x_{n_k}, x) \leqslant \frac{1}{n} + d(x_{n_k}, x),$$

hence $y_{n_k} \longrightarrow x$. *Since* f *is continuous, there exists* $\delta > 0$ *such that:*

$$d_{\mathbb{X}}(x_{n_k}, x) < \delta \Longrightarrow d_{\mathbb{Y}}(f(x_{n_k}), f(x)) < \frac{\varepsilon}{2},$$

$$d_{\mathbb{X}}(y_{n_k}, x) < \delta \Longrightarrow d_{\mathbb{Y}}(f(y_{n_k}), f(x)) < \frac{\varepsilon}{2}.$$

Consequently, we obtain:

$$d_{\mathbb{Y}}(f(x_{n_k}), f(y_{n_k})) \leqslant d_{\mathbb{Y}}(f(x_{n_k}), f(x)) + d_{\mathbb{Y}}(f(x), f(y_{n_k})) < \varepsilon,$$

which contradicts the hypotheses.

Using the previous lemma, we obtain the following result.

Corollaire 4.2. *If* $f : [a,b] \to \mathbb{R}$ *is a continuous function, then* f *is uniformly continuous on* $[a,b]$.

Proposition 4.10. *Let* $(\mathbb{X}, d)$ *be a metric space. Then, we have:*

1. *Every relatively compact subset is precompact.*
2. *If* $(\mathbb{X}, d)$ *is complete, then every precompact subset is relatively compact.*

Proof.

1. *Let A be a relatively compact subset of $\mathbb{X}$; then $Cl(A)$ is precompact, and hence A is precompact.*

2. *Suppose $(\mathbb{X}, d)$ is complete. If A is precompact, then $Cl(A)$ is also precompact. Furthermore, $Cl(A)$ is closed in $\mathbb{X}$, which is complete, and thus $Cl(A)$ is complete as well, showing that $Cl(A)$ is compact (see Proposition (4.8)).*

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