وزارة التعليــم العالــي والبحــث العلمــي

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INTRODUCTION TO METRIC & TOPOLOGICAL SPACES

FOR THE SECOND YEAR LMD MATHEMATICS STUDENTS

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CONTENTS

CHAPTER 5

CONNECTED SPACES

5.1 Connectivity in Topological Spaces

5.1.1 Connected Spaces and Subsets

Let the two spaces $(\mathbb{X}, |.|)$ and $(\mathbb{Y}, |.|)$ be such that $\mathbb{X} =]2,3[\cup]4,5[$ and $\mathbb{Y} = [2,3]\cup]3,4[$. The two subsets $O_1 =]2,3[$ and $O_2 =]4,5[$ are both open and closed in $\mathbb{X}$ because $O_1 = \mathbb{X}\cap]2,3[= \mathbb{X}\cap[2,3]$ and $O_2 = \mathbb{X}\cap]4,5[= \mathbb{X}\cap[4,5]$. Moreover, we have $\mathbb{X} = O_1 \cup O_2$, so the family $\{O_1, O_2\}$ is a partition of $\mathbb{X}$ into two disjoint open (and closed) sets. In this case, we say that $\mathbb{X}$ is not connected, whereas $\mathbb{Y}$ is connected because it can be written in the form $\mathbb{Y} = [2,4[$. The concept of connectivity, which we will define below, intuitively means that a space is "in one piece" or that it cannot be split into two "separated" parts.

Definition 5.1. *Let $(\mathbb{X}, \mathcal{T})$ a topological space. $\mathbb{X}$ is said to be disconnected if it is the union of two disjoint non-empty open sets. In other words, a space is connected if it does not have a partition consisting of two non-empty open sets. We write then,*

$$\mathbb{X} \text{ is connected} \iff \begin{cases} \text{There do not exist } O_1, O_2 \in \mathcal{T} \text{ such that:} \\ \bullet\ O_1 \cup O_2 = \mathbb{X}, \\ \bullet\ O_1 \cap O_2 = \emptyset, \\ \bullet\ O_1 \neq \emptyset \text{ and } O_2 \neq \emptyset. \end{cases}$$

An equivalent definition of the connectivity of $\mathbb{X}$ is as follows.

Definition 5.2. *$\mathbb{X}$ is connected if for any partition of $\mathbb{X}$ into two open sets O_1 and O_2, we have $O_1 = \emptyset$ or $O_2 = \emptyset$.*

Proposition 5.1. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. The following assertions are equivalent.*

1. *$\mathbb{X}$ is connected.*
2. *There does not exist a partition of $\mathbb{X}$ into two non-empty open sets.*
3. *There does not exist a partition of $\mathbb{X}$ into two non-empty closed sets.*
4. *$\emptyset$ and $\mathbb{X}$ are the only sets that are both open and closed (clopen sets) in $\mathbb{X}$.*
5. *Any subset $A \subset \mathbb{X}$ such that $A \neq \emptyset$ and $A \neq \mathbb{X}$ has a non-empty boundary.*
6. *There is no continuous and surjective map from $\mathbb{X}$ to a discrete space $\mathbb{Y}$ containing two elements.*
7. *Every continuous map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ is constant.*

Proof.

1. $\Longrightarrow$ ***2.*** *By definition.*

2. $\Longrightarrow$ ***3.*** *Suppose there exists a partition of $\mathbb{X}$ into two non-empty closed sets F_1 and F_2, i.e., $F_1 \cup F_2 = \mathbb{X}$ and $F_1 \cap F_2 = \emptyset$. Then F_1 and F_2 are two non-empty open sets that form a partition of $\mathbb{X}$ because $\complement_{\mathbb{X}} F_1 = F_2$ and $\complement_{\mathbb{X}} F_2 = F_1$.*

3. $\Longrightarrow$ ***4.*** *Suppose there exists a set A that is both open and closed, and different from $\mathbb{X}$ and $\emptyset$. We deduce that A and $\complement_{\mathbb{X}} A$ form a partition of $\mathbb{X}$ into two non-empty closed sets.*

4. $\Longrightarrow$ ***5.*** *Suppose A is a subset of $\mathbb{X}$ such that $A \neq \emptyset$, $A \neq \mathbb{X}$, and $Cl(A) = \emptyset$. We deduce that A is both open and closed.*

5. $\Longrightarrow$ ***6.*** *Suppose there exists a continuous and surjective map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$. Then, the set $\{a\}$ is both open and closed. Thus, $f^{-1}(\{a\})$ is a set that is both open and closed, such that $f^{-1}(\{a\}) \neq \emptyset$ and $f^{-1}(\{a\}) \neq \mathbb{X}$. Moreover, $Cl(f^{-1}(\{a\})) = \emptyset$.*

6. $\Longrightarrow$ ***7.*** *Suppose there exists a continuous map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ that is not constant. Then f is surjective.*

7. $\Longrightarrow$ ***1.*** *Suppose $\mathbb{X}$ is not connected. Then there exist two non-empty open sets $O_1, O_2 \subset \mathbb{X}$ such that $O_1 \cup O_2 = \mathbb{X}$ and $O_1 \cap O_2 = \emptyset$. Then, the map $f : \mathbb{X} \longrightarrow \mathbb{Y} = \{a, b\}$ defined by $f(x) = a$ if $x \in O_1$ and $f(x) = b$ if $x \in O_2$ is continuous but not constant.*

Example 5.1.

1. *$\mathbb{R}$ is connected.*

2. *Any discrete space $(\mathbb{X}, \delta)$ such that card$(\mathbb{X}) \geqslant 2$ is not connected. Indeed, if $x \in \mathbb{X}$, then we have $\{x\} \cup \complement_{\mathbb{X}}\{x\} = \mathbb{X}$ and $\{x\} \cap \complement_{\mathbb{X}}\{x\} = \emptyset$, with $\{x\}$ and $\complement_{\mathbb{X}}\{x\}$ being two open (two closed) sets.*

3. *It is evident that any space equipped with the trivial topology is connected.*

4. *Let $\mathbb{X} = \{a, b, c, d\}$ and $\mathcal{T} = \{\emptyset, \mathbb{X}, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. It is clear that $(\mathbb{X}, \mathcal{T})$ is connect.*

Definition 5.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A a non-empty subset of $\mathbb{X}$. We say that A is connected if the subspace $(A, \mathcal{T}_A)$ is connected. Classically, we consider the empty set as connected.*

Example 5.2.

1. *Every interval in $\mathbb{R}$ is connected.*

2. *Every open (closed) ball in $\mathbb{R}^n$ is connected.*

3. *The space $(\mathbb{R}^*, |\cdot|)$ is not connected (why ?).*

5.1.2 Properties of Connected Spaces

Proposition 5.2. *If a subset A of a topological space $(\mathbb{X}, \mathcal{T})$ is connected, then the existence of two open sets $O_1, O_2 \in \mathcal{T}$ such that $A \subset O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$ implies that $A \subset O_1$ or $A \subset O_2$.*

Proof. *Suppose A is connected and let $O_1, O_2 \in \mathcal{T}$ such that $A \subset O_1 \cup O_2$ and $O_1 \cap O_2 = \emptyset$. Then, $A = (A \cap O_1) \cup (A \cap O_2)$ and $(A \cap O_1) \cap (A \cap O_2) = \emptyset$. Since A is connected, we obtain $(A \cap O_1 = \emptyset)$ or $(A \cap O_2 = \emptyset)$, from which it follows that $A \subset O_2$ or $A \subset O_1$.*

Proposition 5.3. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A, B two subsets of $\mathbb{X}$ such that A is connected and $A \subset B \subset Cl(A)$. Then, we have:*

1. *If A is connected, then B is connected.*

2. *If A is connected, then $Cl(A)$ is connected.*

3. *If A is a connected and dense subset of $\mathbb{X}$, then $\mathbb{X}$ is connected.*

Proof.

1. *Let $f : B \longrightarrow \{0,1\}$ be a continuous function. Since A is connected and f is continuous on A, we obtain that f is constant on A. Since f is continuous on B, the set $G = \{x \in B : f(x) \in f(A)\}$ is a closed set of B containing A, so $Cl(A)_B \subset G$. Thus, f is constant on the closure of A in B, which is $Cl(A)_B = B \cap Cl(A) = B$. We conclude that f is constant on B. Therefore, B is connected.*

2. *It is sufficient to take $B = Cl(A)$ in **(1)**).*

3. *We have $Cl(A) = \mathbb{X}$ because A is dense in $\mathbb{X}$, and $Cl(A)$ is connected because A is connected (see question **(2)**). We conclude that $\mathbb{X}$ is connected.*

Proposition 5.4. *Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \longrightarrow \mathbb{Y}$ be a continuous function. If $\mathbb{X}$ is connected, then $f(\mathbb{X})$ is a connected subset of $\mathbb{Y}$.*

Proof. *Let G be a subset of $f(\mathbb{X})$ that is both open and closed in the induced topology. Since f is continuous as a function with values in $f(\mathbb{X})$, we deduce that $f^{-1}(G)$ is both open and closed in $\mathbb{X}$. Since $\mathbb{X}$ is connected, we deduce that $f^{-1}(G) = \emptyset$ or $f^{-1}(G) = \mathbb{X}$. Since $f(f^{-1}(G)) = G$, we obtain that $G = \emptyset$ or $G = f(\mathbb{X})$, which shows that $f(\mathbb{X})$ is connected.*

Remark 5.1. *According to the previous proposition, connectedness is a topological property.*

Proposition 5.5. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space.*

1. *If $\{A_i : i \in I\}$ is an arbitrary family of connected subsets of $\mathbb{X}$ such that $\bigcap_{i\in I} A_i \neq \emptyset$, then $\bigcup_{i\in I} A_i$ is connected.*

2. *If $\{A_i : i \in I\}$ is an arbitrary family of connected subsets of $\mathbb{X}$ such that $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$, then $\bigcup_{i\in I} A_i$ is connected.*

3. *If $\{A_i : i \in I\}$ is an arbitrary totally ordered family of connected subsets of $\mathbb{X}$, then $\bigcup_{i\in I} A_i$ is connected.*

We will provide the proof for the first case only.

Proof. *Let* $a \in \bigcap_{i\in I} A_i \neq \emptyset$. *If* $f : \bigcup_{i\in I} A_i \longrightarrow \{0,1\}$ *is a continuous function, then* $f_{|A_i}$ *is continuous, and thus constant by the connectedness of* A_i. *Since* $a \in A_i$ *for all* $i \in I$, *we obtain* $f(x) = f(a)$ *for all* $x \in A_i$. *Therefore,* $f(x) = f(a)$ *for all* $x \in \bigcup_{i\in I} A_i$, *i.e.,* f *is constant on* $\bigcup_{i\in I} A_i$, *which shows that* $\bigcup_{i\in I} A_i$ *is connected.*

Proposition 5.6. *A subset* A *of* $\mathbb{R}$ *is connected if and only if it is an interval.*

Proof.

$\Longrightarrow$) *Suppose that the set* $A \subset \mathbb{R}$ *is not an interval in* $\mathbb{R}$. *Then, there exist points* $x, y \in A$ *and* $z \notin A$ *such that* $x < z < y$. *Define* $O_1 =]-\infty, z[\cap A$ *and* $O_2 =]z, +\infty[\cap A$, *which are two non-empty open subsets of* A. *Furthermore, we have* $O_1 \cap O_2 = \emptyset$ *and* $O_1 \cup O_2 = A$. *Therefore,* A *is not connected.*

$\Longleftarrow$) *Let* A *be a non-empty interval in* $\mathbb{R}$. *Suppose* $A = O_1 \cup O_2$, *where* O_1 *and* O_2 *are two non-empty open subsets of* A *with* $O_1 \cap O_2 = \emptyset$. *Let* $x \in O_1$ *and* $y \in O_2$ *such that* $x < y$, *and let* $z = \sup(O_1 \cap [x,y])$.

On the one hand, if $z \in O_1$, *then* $z < y$, *which implies the existence of a real number* $r > 0$ *such that* $[z, z+r[\subset O_1 \cap [x,y]$, *contradicting the definition of* z.

On the other hand, if $z \in O_2$, *then* $z > x$, *which implies the existence of a real number* $r > 0$ *such that* $]z-r, z] \subset O_2 \cap [x,y]$, *again contradicting the definition of* z.

Thus, we conclude that $z \notin O_1$ *and* $z \notin O_2$, *which is impossible because* $[x,y] \subset A$. *Therefore,* A *is connected.*

Proposition 5.7. *Let* $(\mathbb{X}, \mathcal{T})$ *be a topological space and* $f : \mathbb{X} \longrightarrow \mathbb{R}$ *a continuous function.*

1. *The image of any connected subset of* $\mathbb{X}$ *is an interval in* $\mathbb{R}$.
2. *Let* $a, b \in f(\mathbb{X})$. *If* $\mathbb{X}$ *is connected, then the equation* $f(x) = c$ *has a solution for every* $c \in [a,b]$.

Proof.

1. *Let A be a connected subset of X. Then, $f(A)$ is connected in $\mathbb{R}$ (see Proposition 5.4), which implies that $f(A)$ is an interval (see Proposition 5.6).*

2. *Using the two propositions (5.4) and (5.6), we conclude that $f(\mathbb{X})$ is an interval. Then, $[a,b] \subset f(\mathbb{X})$ which implies that*

$$\forall c \in [a,b], \quad c \in f(\mathbb{X}).$$

Therefore, there exists $x \in \mathbb{X}$ such that $f(x) = c$.

Proposition 5.8. *Let $((\mathbb{X}, \mathcal{T}_{\mathbb{X}}))$ and $((\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces. Then $\mathbb{X} \times \mathbb{Y}$ is connected if and only if $\mathbb{X}$ and $\mathbb{Y}$ are connected.*

Proof.

$\Longrightarrow$) *Suppose that $\mathbb{X} \times \mathbb{Y}$ is connected. We have $p_{\mathbb{X}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{X}$ and $p_{\mathbb{Y}}(\mathbb{X} \times \mathbb{Y}) = \mathbb{Y}$, where $p_{\mathbb{X}}$ and $p_{\mathbb{Y}}$ are the continuous canonical projections. It follows that $\mathbb{X}$ and $\mathbb{Y}$ are connected.*

$\Longleftarrow$) *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ are connected, and let $f : \mathbb{X} \times \mathbb{Y} \longrightarrow \{0,1\}$ be a continuous function. Then, it suffices to show that f is constant. Since $\mathbb{Y}$ is connected, the function $f(x,\cdot) : \mathbb{Y} \longrightarrow \{0,1\}$ is constant, meaning $f(x,y_1) = f(x,y_2)$ for all $x \in \mathbb{X}$.*

Since $\mathbb{X}$ is connected, the function $f(\cdot,y) : \mathbb{X} \longrightarrow \{0,1\}$ is constant, meaning $f(x_1,y) = f(x_2,y)$ for all $y \in \mathbb{Y}$. Therefore, $f(x_1,y_1) = f(x_2,y_2)$ for all $(x_1,y_1),(x_2,y_2) \in \mathbb{X} \times \mathbb{Y}$, which shows that f is constant. Thus, $\mathbb{X} \times \mathbb{Y}$ is connected.

In the general case, we have the following result.

Proposition 5.9. *Let $\{(\mathbb{X}_i, \mathcal{T}_i) : i \in I\}$ be a family of topological spaces. Then $\prod_{i \in I} \mathbb{X}_i$ is connected if and only if $\mathbb{X}_i$ is connected for every $i \in I$.*

5.1.3 Connected components, locally connected spaces

Definition 5.4. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. For each $x \in \mathbb{X}$, we call the connected component of x, denoted by $\mathcal{C}(x)$, the equivalence class of x under the relation $\mathcal{R}$ defined* by " $x\mathcal{R}y \Longleftrightarrow x$ and y belong to the same connected subset of $\mathbb{X}$."

Remark 5.2. *According to the previous definition, we conclude that the connected component of a point x is the union of all connected subsets containing x. In other words, it is the largest connected subset containing x. Moreover, the connected components of $\mathbb{X}$ form a partition of $\mathbb{X}$.*

Definition 5.5. *A connected component of a space $\mathbb{X}$ is a maximal connected subset of $\mathbb{X}$, i.e., a connected subset that is not contained in any other (strictly) larger connected subset of X.*

Example 5.3.

1. *The only connected component in $(\mathbb{R}, |\cdot|)$ is $\mathbb{R}$ itself.*
2. *$(\mathbb{R}^*, |\cdot|)$ has two connected components: $\mathbb{R}^*_-$ and $\mathbb{R}^*_+$.*

Definition 5.6. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subset X$. The connected components of A are defined as the connected components of $(A, \mathcal{T}_A)$.*

Proposition 5.10. *Every connected component is closed.*

Proof. *Let A be a connected component. Then A is connected, and thus $Cl(A)$ is a connected subset containing A, so $Cl(A) = A$, which shows that A is closed.*

Definition 5.7. *Let $(\mathbb{X}, \mathcal{T})$ be a topological space. We say that $\mathbb{X}$ is locally connected if every point $x \in \mathbb{X}$ admits a neighborhood basis consisting of open connected sets.*

Example 5.4.

1. *$\mathbb{R}$ is locally connected.*
2. *$\mathbb{Q}$ is not locally connected.*
3. *Every discrete space is locally connected. Indeed, $\mathcal{N}(x) = \{\{x\}\}$ forms a neighborhood basis consisting of open connected sets for each point $x \in \mathbb{X}$.*

Proposition 5.11. *Let $(\mathbb{X},\mathcal{T})$ be a topological space. X is locally connected if and only if every connected component of every open set in $\mathbb{X}$ is open.*

Proof.

$\Longrightarrow$) *Suppose that $\mathbb{X}$ is locally connected. Let O be an open set in $\mathbb{X}$, and let $\mathcal{C}(O)$ be a connected component of O. Then, for every $x \in \mathcal{C}(O)$, there exists $N \in \mathcal{N}(x)$ such that N is connected and $N \subset O$. Thus, $N \subset \mathcal{C}(O)$, which shows that $\mathcal{C}(O)$ is open (a neighborhood of each of its points).*

$\Longleftarrow$) *Let $x \in X$ and N be an open neighborhood of x. Then, the connected component of x in N is open, which shows that X is locally connected.*

5.1.4 Path-connectedness

Definition 5.8. *Let $(X,\mathcal{T})$ be a topological space and $[x,y]$ an interval in $\mathbb{R}$. A path in a subset A of X is any continuous function $\gamma : [x,y] \longrightarrow A$. The image $\gamma([x,y])$ of the path is called an arc with starting point $\gamma(x)$ and endpoint $\gamma(y)$.*

Remark 5.3. *We can replace $[x,y]$ with $[0,1]$ because they are homeomorphic.*

Definition 5.9. *Let $(\mathbb{X},\mathcal{T})$ be a topological space and A a subset of X. We say that A is arc-connected if for every $a,b \in A$, there exists an arc contained in A with starting point a and endpoint b.*

Example 5.5.

1. *$\mathbb{R}$ is arc-connected. It is enough to take as a path in $\mathbb{R}$ the map $\gamma : [0,1] \longrightarrow \mathbb{R}$ defined by $\gamma(x) = a + x(b-a)$, for all $a,b \in \mathbb{R}$.*
2. *$\mathbb{Q}$ and $\complement_{\mathbb{R}}\mathbb{Q}$ are not arc-connected.*

Proposition 5.12. *An arc-connected space is connected.*

Proof. *Suppose that $\mathbb{X}$ is an arc-connected space and let $a \in \mathbb{X}$. Then, for every $b \in \mathbb{X}$, there exists a continuous function $\gamma_b : [0,1] \to \mathbb{X}$ such that $\gamma_b(0) = a$ and $\gamma_b(1) = b$. Therefore, the collection $\{\gamma_b([0,1]) : b \in \mathbb{X}\}$ forms a family of connected sets whose intersection is non-empty (since it contains a), and $\mathbb{X} = \bigcup_{b\in\mathbb{X}} \gamma_b([0,1])$ which implies that $\mathbb{X}$ is connected.*

5.2 Connectedness in Metric Spaces

The definitions and properties of connectedness in metric spaces are the same as those we have seen in topological spaces. Therefore, it is enough to give a brief reminder of these definitions and properties.

5.2.1 Definitions and properties of connectivity in metric spaces

- $\mathbb{X}$ is connected if and only if the only subsets of $\mathbb{X}$ that are both open and closed are the empty set $\emptyset$ and $\mathbb{X}$.
- $\mathbb{X}$ is connected if and only if there is no partition of $\mathbb{X}$ into two non-empty open sets.
- $\mathbb{X}$ is connected if and only if there is no partition of $\mathbb{X}$ into two non-empty closed sets.
- $\mathbb{X}$ is connected if and only if every continuous function $f : (\mathbb{X}, d) \longrightarrow (\{0,1\}, \delta)$ is constant.
- The continuous image of a connected set is connected.
- Connectivity is a topological property.
- $\mathbb{X} \times \mathbb{Y}$ is connected if and only if both $\mathbb{X}$ and $\mathbb{Y}$ are connected.
- If A is connected and $A \subset B \subset Cl(A)$, then B is connected.
- If A is connected, then $Cl(A)$ is also connected.
- $\mathbb{X}$ is arc-connected if for all $a, b \in \mathbb{X}$, there exists a continuous function $f : [0,1] \longrightarrow \mathbb{X}$ such that $f(0) = a$ and $f(1) = b$.
- Every arc-connected space is connected.
- A connected space is not necessarily arc-connected.
- A subset $A \subset \mathbb{R}$ is connected if and only if A is an interval.
- If $\mathbb{X}$ is connected and $f : \mathbb{X} \longrightarrow \mathbb{R}$ is continuous, then $f(\mathbb{X})$ is an interval.
- If $f : [a,b] \longrightarrow \mathbb{R}$ is continuous, then $f([a,b]) = [c,d]$.

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