وزارة التعليـم العالـي والبحـث العلمـي
Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 3: Continuity in Metric Spaces

Exercise 1:

1. Find $d(3,]3,5[)$ and $d(\sqrt{2}, \mathbb{Q})$.
2. Let $A = \{(x,y) \in \mathbb{R}^2 : xy = 1\}$ and $B = \{(x,y) \in \mathbb{R}^2 : y = 0\}$, calculate $d(A,B)$.
3. Calculate $\text{diam}([1,2) \cap \mathbb{Q})$ and $\text{diam}([-2,1) \cap (\complement_{\mathbb{R}}\mathbb{Q})$.

Exercise 2:

1. Show that (a,b) and (c,d), where $a,b,c,d \in \mathbb{R}$, are homeomorphic.
2. Show that $\mathbb{R}$ and $(-1,1)$ are homeomorphic.
3. Is $\mathbb{R}$ homeomorphic to every open interval in $\mathbb{R}$?

Exercise 3: Let $(\mathbb{X}, d)$ be a metric space and $f : (\mathbb{X}, d) \longrightarrow (\mathbb{R}, |\cdot|)$. Show that f is continuous if and only if, for every $a \in \mathbb{R}$, the sets $f^{-1}((a, +\infty))$ and $f^{-1}((-\infty, a))$ are open in $\mathbb{X}$.

Exercise 4: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q}$ (the set of rational numbers). Prove that $f(x) = 0$ for all $x \in \mathbb{R}$ (the set of real numbers).

Exercise 5: Let $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$ be continuous functions. Show that if $g \circ f$ is a homeomorphism and f is surjective, then f and g are homeomorphism.

Exercise 6: Let $(\mathbb{X}, d)$ be a metric space, and let A and B be two non-empty, closed subsets of $\mathbb{X}$ such that $A \cap B = \emptyset$. Define the functions $f, g : \mathbb{X} \to \mathbb{R}_+$ by

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}, \quad g(x) = d(x,A).$$

1. Show that for any continuous function $h : \mathbb{X} \to \mathbb{R}$, the set $K = \{x \in \mathbb{X} : h(x) = 0\}$ is closed.
2. Show that g is continuous.
3. Deduce that the function f is continuous.
4. Find $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$.