وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
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Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 3: Continuity in Metric Spaces

Exercise 1:

1. We have $3 \in Cl((3,5)) = [3,5]$, which implies $d(3,(3,5)) = 0$.

   We have $\sqrt{2} \in Cl(\mathbb{Q}) = \mathbb{R}$, which implies $d(\sqrt{2}, \mathbb{Q}) = 0$.

2. $d(A,B) = 0$ because,

$$\lim_{n \to +\infty} \left(\frac{1}{x}\right) = 0.$$

3.

$$\begin{aligned} diam\left([1,2) \cap \mathbb{Q}\right) &= \sup_{x,y \in [1,2) \bigcap \mathbb{Q}} d(x,y) = |2-1| = 1. \\ diam\left([-2,1) \cap \complement_{\mathbb{R}} \mathbb{Q}\right) &= \sup_{x,y \in [-2,1) \bigcap \complement_{\mathbb{R}} \mathbb{Q}} d(x,y) = |1-(-2)| = 3. \end{aligned}$$

Exercise 2:

1. Let $f : (a,b) \to (c,d)$ be defined by

$$f(x) = c + (d-c)\frac{x-a}{b-a},$$

for all $a,b,c,d \in \mathbb{R}$. It is clear that f is a continuous and bijective function (because it is a polynomial). Its inverse $f^{-1} : (c,d) \to (a,b)$ is given by

$$f^{-1}(x) = a + (b-a)\frac{x-c}{d-c},$$

for all $a,b,c,d \in \mathbb{R}$. It is also clear that f^{-1} is continuous (because it is a polynomial). Hence, f is a homeomorphism, and therefore, (a,b) and (c,d) are homeomorphic.

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2. Let $f : \mathbb{R} \to (-1,1)$ be defined by

$$f(x) = \frac{x}{1+|x|},$$

for all $x \in \mathbb{R}$. It is clear that f is continuous and bijective. Its inverse $f^{-1} : (-1,1) \to \mathbb{R}$ is given by

$$f^{-1}(x) = \frac{x}{1-|x|}.$$

It is also clear that f^{-1} is continuous. Hence, f is a homeomorphism, and therefore, $\mathbb{R}$ and $(-1,1)$ are homeomorphic.

3. Since the relation *homeomorphic* is transitive, $\mathbb{R}$ is homeomorphic to any open interval (using parts (1) and (2): $\mathbb{R} \xrightarrow{\text{homeo} \dots} (-1,1) \xrightarrow{\text{homeo} \dots} (c,d)$).

Exercise 3: Let $(\mathbb{X}, d)$ be a metric space and $f : (\mathbb{X}, d) \to (\mathbb{R}, |\cdot|)$.

$\Longrightarrow$) Suppose that f is continuous. Since the sets $(-\infty, a)$ and $(a, +\infty)$ are open in $\mathbb{R}$, it follows that $f^{-1}((-\infty, a))$ and $f^{-1}((a, +\infty))$ are open in $\mathbb{X}$.

$\Longleftarrow$) Suppose that $f^{-1}((-\infty, a))$ and $f^{-1}((a, +\infty))$ are open in $\mathbb{X}$. Let $a < b$. Then

$$(a,b) = (-\infty, b) \cap (a, +\infty).$$

Therefore,

$$f^{-1}((a,b)) = f^{-1}((-\infty, b)) \cap f^{-1}((a, +\infty))$$

is open in $\mathbb{X}$ (since it is the intersection of two open sets).

Now, let O be an open set in $\mathbb{R}$. Then $O = \bigcup_{i \in I} (a_i, b_i)$, where (a_i, b_i) are open intervals. It follows that

$$f^{-1}(O) = \bigcup_{i \in I} f^{-1}((a_i, b_i))$$

is open in $\mathbb{X}$ (since it is the union of open sets). This implies that f is continuous.

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Exercise 4: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$f(x) = 0, \ \textit{for all } x \in \mathbb{Q}. \tag{i}$$

We need to show that

$$f(x) = 0, \ \textit{for all } x \in \complement_{\mathbb{R}}\mathbb{Q}. \tag{ii}$$

Let $x \in \complement_{\mathbb{R}}\mathbb{Q}$. Then $x \in Cl(\mathbb{Q}) = \mathbb{R}$. Therefore, there exists a sequence $(x_n) \subset \mathbb{Q}$ such that $x_n \to x$ as $n \to +\infty$. Since f is continuous, we have $f(x_n) \to f(x)$. But by the hypothesis, $f(x_n) = 0$ (because $x_n \in \mathbb{Q}$), which implies $f(x) = 0$ for all $x \in \complement_{\mathbb{R}}\mathbb{Q}$.

Finally, combining (i) and (ii), we conclude that $f(x) = 0$ for all $x \in \mathbb{R}$.

Exercise 5: Let $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$ be continuous functions. Show that if $g \circ f$ is a homeomorphism and f is surjective, then f and g are homeomorphism.

Exercise 6: Let $(\mathbb{X}, d)$ be a metric space, and let A and B be two non-empty, closed subsets of $\mathbb{X}$ such that $A \cap B = \emptyset$. Define the functions $f, g : \mathbb{X} \to \mathbb{R}_+$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad g(x) = d(x, A).$$

1. The set $K = \{x \in \mathbb{X} : h(x) = 0\} = h^{-1}(\{0\})$ is closed because it is the preimage of a closed set in $\mathbb{R}$ under a continuous function.

2. Let $a \in A$. Then

$$d(x, a) \leq d(x, y) + d(y, a), \quad \forall x, y \in A,$$

which shows that

$$d(x, A) - d(y, A) \leqslant d(x, y), \quad \forall x, y \in A. \tag{i}$$

Using similar arguments, we also obtain

$$d(y, A) - d(x, A) \leqslant d(x, y), \quad \forall x, y \in A. \tag{ii}$$

From (i) and (ii), we deduce

$$|g(x) - g(y)| = |d(x, A) - d(y, A)| \leqslant d(x, y),$$

which shows that g is continuous (1-Lipschitz).

3. Since A and B are closed, $x \in Cl(A) = A \implies d(x, A) = 0$. Moreover, since $A \cap B = \emptyset$, we deduce that $x \notin B = Cl(B)$, and thus $d(x, B) > 0$. This shows that f is well-defined because $d(x, A)$ and $d(x, B)$ cannot both be zero simultaneously. Now, since $x \mapsto d(x, A)$ and $x \mapsto d(x, B)$ are continuous functions (as shown earlier), the function f is continuous because it is the quotient of two continuous functions.

4. We have

$$\begin{aligned} f(x) = 0 &\iff d(x, A) = 0 \\ &\iff x \in Cl(A) = A. \end{aligned}$$

Thus, $A = \{x \in \mathbb{X} : f(x) = 0\} = f^{-1}(\{0\})$.

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5. Similarly,

$$\begin{aligned} f(x) = 1 &\iff d(x, B) = 0 \\ &\iff x \in Cl(B) = B. \end{aligned}$$

Thus, $B = \{x \in \mathbb{X} : f(x) = 1\} = f^{-1}(\{1\})$.

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