وزارة التعليـم العالـي والبحـث العلمـي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 4: Complete metric spaces

Exercise 1: Among the following sets, indicate those that are complete in the space $(\mathbb{R}, |.|)$.

1) $\mathbb{Q}$, **2)** $\complement_{\mathbb{R}}\mathbb{Q}$, **3)** $\mathbb{Q} \cap [5, 6]$, **4)** $(1, +\infty)$, **5)** $\mathbb{N}$, **6)** $[a, b]$, such that $a, b \in \mathbb{R}$.

Exercise 2: Show that every discrete metric space is complete.

Exercise 3: Is $\mathbb{R}$ complete with the following distances:

1) $d_a(x, y) = |x^3 - y^3|$, **2)** $d_b(x, y) = |e^x - e^y|$, **3)** $d_c(x, y) = |\arctan(x) - \arctan(y)|$.

Exercise 4: Let $(\mathbb{X}, d)$ be a complete metric space and $\mathbb{Y} \subset \mathbb{X}$. Show that $(\mathbb{Y}, d)$ is a complete space if and only if $\mathbb{Y}$ is closed set in $\mathbb{X}$.

Exercise 5: Let $(\mathbb{X}, d)$ be a metric space. Show that any finite union of complete subsets of $\mathbb{X}$ is a complete subset of $\mathbb{X}$.

Exercise 6: Let $(\mathbb{X}, d_{\mathbb{X}})$ and $(\mathbb{Y}, d_{\mathbb{Y}})$ be two metric spaces such that $(\mathbb{X}, d_{\mathbb{X}})$ is complete. Let A be a closed set, and $f : A \longrightarrow \mathbb{Y}$ a continuous function that satisfies $d_{\mathbb{Y}}(f(x), f(y)) \geqslant d_{\mathbb{X}}(x, y)$ for all $x, y \in A$. Show that $f(A)$ is closed in $(\mathbb{Y}, d_{\mathbb{Y}})$.

Exercise 7: Let $f : (\mathbb{X}, d_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, d_{\mathbb{Y}})$ be a uniformly continuous function, and let $(x_n)_n$ be a Cauchy sequence in $\mathbb{X}$.

1. Show that $(f(x_n))$ is a Cauchy sequence in $\mathbb{Y}$.
2. If, in addition, f is bijective and f^{-1} is continuous, show that if $\mathbb{Y}$ is complete, then $\mathbb{X}$ is complete.

Exercise 8: Let $\mathbb{X} = \{x \in \mathbb{Q} : x \geqslant 1\}$. Consider the function $f : \mathbb{X} \longrightarrow \mathbb{X}$ defined by

$$f(x) = \frac{x}{2} + \frac{1}{x}$$

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1. Show that $\forall x, y \in \mathbb{X}$, $|f(x) - f(y)| \leqslant \frac{1}{2}|x - y|$.
2. Show that f has no fixed points.
3. Why does the result of the previous question not contradict the fixed-point theorem?

Exercise 9:

1. Show that for $x \geq 1$ and $t \geq 0$ we have $\sqrt{x+t} - \sqrt{x} \leq \frac{t}{2}$.
2. Show that $f(x) = \sqrt{x}$ is a contraction on $[1, +\infty)$.
3. Find the fixed point of f.

Exercise 10: Let $f : [1, +\infty) \longrightarrow [1, +\infty)$ be defined by $f(x) = x + \frac{1}{x}$.

1. Show that $[1, +\infty)$ is complete and that $|f(x) - f(y)| < |x - y|$ for all $x, y \in [1, +\infty)$.
2. Show that f has no fixed points.