وزارة التعليم العالي والبحث العلمي

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Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solutions of series 4: Complete metric spaces

Exercise 1:

1. $\mathbb{Q}$ is not complete because it is not closed. We can also see that the sequence $\left(1+\frac{1}{n}\right)^n$ of $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{R}$, and hence in $\mathbb{Q}$, but its limit $e \notin \mathbb{Q}$.
2. $\complement_{\mathbb{X}}\mathbb{Q}$ is not complete because it is not closed.
3. $\mathbb{Q} \cap [5, 6]$ is not complete because it is not closed.
4. $(1, +\infty)$ is not complete because it is not closed.
5. The set $\mathbb{N}$ is closed in $\mathbb{R}$ (which is complete), because its complement is an infinite union of open sets and thus is open. This implies that $\mathbb{N}$ is complete.
6. The set $[a, b]$ is closed in $\mathbb{R}$ (which is complete). Therefore it is complete.

Exercise 2: Let (x_n) be a Cauchy sequence in a discrete metric space $(\mathbb{X}, \delta)$. By definition of a Cauchy sequence, we have:

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n, m \in \mathbb{N},\ n, m \geqslant n_0 \implies \delta(x_n, x_m) < \varepsilon.$$

In a discrete metric space, $\delta(x_n, x_m) < \varepsilon$ implies $x_n = x_m$ whenever $\varepsilon < 1$. For instance, if $\varepsilon = \dfrac{1}{4}$, we find $x_n = x_m$ for all $n, m \geqslant n_0$.

Thus, (x_n) is constant from n_0 onward, meaning it converges. Therefore, the metric space $(\mathbb{X}, \delta)$ is complete.

Exercise 3:

1. Let $d_a(x, y) = |x^3 - y^3|$. Is $(\mathbb{R}, d_a)$ complete?

   Let (x_n) be a d_a-Cauchy sequence, so $d_a(x_n, x_m) = |x_n^3 - x_m^3| \to 0$ as $n, m \to +\infty$. Thus, (x_n^3) is a d_u-Cauchy sequence in $(\mathbb{R}, d_u)$, which is complete, implying that (x_n^3) is convergent in $(\mathbb{R}, d_u)$. Therefore, there exists $y \in \mathbb{R}$ such that $|x_n^3 - y| \to 0$ as $n \to +\infty$. Let $x = y^{1/3}$, then $x_n^3 \to x^3 \in \mathbb{R}$, i.e., $d_a(x_n, x) = |x_n^3 - x^3| = |x_n^3 - y| \to 0$ as $n \to +\infty$. Thus, (x_n) is d_a-convergent, which shows that $(\mathbb{R}, d_a)$ is a complete space.

2. Let $d_b(x, y) = |e^x - e^y|$. Is $(\mathbb{R}, d_b)$ complete?

   Let $(x_n) = (-n)_{n\in\mathbb{N}}$. We have $d_b(x_n, x_m) = |e^{-n} - e^{-m}| \to 0$ as $n, m \to +\infty$. Thus, (x_n) is a d_b-Cauchy sequence. If we assume that $x_n \to \ell$ in $(\mathbb{R}, d_b)$, then $e^{-n} \to e^{-\ell}$ in $(\mathbb{R}, d_u)$ as $n \to +\infty$. By the uniqueness of limits, we obtain $e^{-\ell} = 0$, which is impossible. Therefore, (x_n) does not converge in $(\mathbb{R}, d_b)$, which shows that the space $(\mathbb{R}, d_b)$ is not complete.

3. Let $d_c(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$. Is $(\mathbb{R}, d_c)$ complete?

   Let $(x_n) = (n)_{n\in\mathbb{N}}$. We have $\tan^{-1}(n) \to \frac{\pi}{2}$ as $n \to +\infty$ in $(\mathbb{R}, d_u)$. Thus, $(\tan^{-1}(x_n))$ is a d_u-Cauchy sequence, which implies that (x_n) is a d_c-Cauchy sequence. If we assume that $x_n \to \ell$ in $(\mathbb{R}, d_c)$, then $\tan^{-1}(n) \to \tan^{-1}(\ell)$ in $(\mathbb{R}, d_u)$ as $n \to +\infty$. By the uniqueness of limits, we obtain $\tan^{-1}(\ell) = \frac{\pi}{2}$ for some $\ell \in \mathbb{R}$, which is impossible. Therefore, (x_n) does not converge in $(\mathbb{R}, d_c)$, which shows that the space $(\mathbb{R}, d_c)$ is not complete.

Exercise 4:

$\Longrightarrow$) Suppose that $\mathbb{Y}$ is complete, and let $x \in Cl(\mathbb{Y})$ (the closure of $\mathbb{Y}$). Then, there exists a sequence (y_n) in $\mathbb{Y}$ that converges to $x \in \mathbb{X}$. Since $\mathbb{Y}$ is complete, the limit must belong to $\mathbb{Y}$. By the uniqueness of the limit, we conclude that $x \in \mathbb{Y}$. Therefore, $\mathbb{Y}$ is closed.

$\Longleftarrow$) Suppose that $\mathbb{Y}$ is closed in $\mathbb{X}$, where $\mathbb{X}$ is complete. Let (y_n) be a Cauchy sequence in $\mathbb{Y} \subset \mathbb{X}$. Since $\mathbb{X}$ is complete, there exists $x \in \mathbb{X}$ such that $y_n \to x$. Because $\mathbb{Y}$ is closed, it must contain all its limit points, so $x \in \mathbb{Y}$. Consequently, $(\mathbb{Y}, d_\mathbb{Y})$ is complete.

Exercise 5: Let $\{A_i : i = 1, 2, \ldots, n\}$ be a family of complete subsets of $(\mathbb{X}, d)$. If (x_n) is a Cauchy sequence in $\bigcup_{i=1}^{n} A_i$, then there exists $i_0 \in \{1, 2, \ldots, n\}$ and a subsequence $(x_{n_k}) \subset A_{i_0}$. Therefore, (x_{n_k}) is a Cauchy sequence in A_{i_0}, which is complete, implying the existence of an element $x \in A_{i_0}$ such that $x_{n_k} \to x$. Consequently, $x_n \to x \in \bigcup_{i=1}^{n} A_i$, showing that $\bigcup_{i=1}^{n} A_i$ is complete.

Exercise 6: It suffices to show that $Cl(f(A)) \subseteq f(A)$.

Let $y \in \text{Adh}(f(A))$. Then, there exists a sequence $(x_n) \subset A$ such that $f(x_n) \to y$. Therefore, $f(x_n)$ is a Cauchy sequence in $\mathbb{Y}$. By hypothesis, we have $d_{\mathbb{Y}}(f(x_n), f(x_m)) \geqslant d_{\mathbb{X}}(x_n, x_m)$ for all $n, m \in \mathbb{N}$. Thus, (x_n) is a Cauchy sequence in $\mathbb{X}$, which is complete. Hence, there exists $x \in \mathbb{X}$ such that $x_n \to x$. Since A is closed, it follows that $x \in A$. Because f is a continuous function, we deduce that $f(x_n) \to f(x)$. By the uniqueness of the limit, we conclude that $y = f(x) \in f(A)$. Thus, $Cl(f(A)) \subseteq f(A)$.

Exercise 7:

1. Since f is uniformly continuous, we have

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x, y \in \mathbb{X},\ d_{\mathbb{X}}(x,y) < \delta \implies d_{\mathbb{Y}}(f(x), f(y)) < \varepsilon. \tag{i}$$

   And since (x_n) is a Cauchy sequence, we have

$$\forall \delta > 0,\ \exists n_0 \in \mathbb{N}\ :\ \forall n, m \in \mathbb{N},\ n, m \geqslant n_0 \implies d_{\mathbb{X}}(x_n, x_m) < \delta. \tag{ii}$$

   From (i) and (ii), we deduce that

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n, m \in \mathbb{N},\ n, m \geqslant n_0 \implies d_{\mathbb{Y}}(f(x_n), f(x_m)) < \varepsilon,$$

   which shows that $(f(x_n))$ is a Cauchy sequence in $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$.

2. Let (x_n) be a Cauchy sequence in $(\mathbb{X}, d_{\mathbb{X}})$. Then, by (1), $(f(x_n))$ is a Cauchy sequence in $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$. By hypothesis, $\mathbb{Y}$ is complete, so there exists $y \in \mathbb{Y}$ such that $f(x_n) \to y$, which implies that $x_n \to f^{-1}(y) \in \mathbb{X}$, since f^{-1} is continuous. Therefore, (x_n) is a convergent sequence in $(\mathbb{X}, d_{\mathbb{X}})$. Consequently, $(\mathbb{X}, d_{\mathbb{X}})$ is a complete space.

Exercise 8: Let $\mathbb{X} = \{x \in \mathbb{Q} : x \geqslant 1\}$. Consider the function $f : \mathbb{X} \longrightarrow \mathbb{X}$ defined by

$$f(x) = \frac{x}{2} + \frac{1}{x}$$

.

1. Let $x, y \in \mathbb{X}$, then $x, y \geqslant 1$.

   We have:

$$\begin{aligned} |f(x) - f(y)| &= \left| \frac{x}{2} + \frac{1}{x} - \frac{y}{2} - \frac{1}{y} \right| \\ &= \left| \frac{1}{2}(x-y) - \frac{x-y}{xy} \right| \\ &= |x-y| \left| \frac{1}{2} - \frac{1}{xy} \right|. \end{aligned}$$

Since $0 < \frac{1}{xy} < 1$, we deduce that $\left|\frac{1}{2} - \frac{1}{xy}\right| \leqslant \frac{1}{2}$, which implies:

$$|f(x) - f(y)| \leqslant \frac{1}{2}|x - y|.$$

2. We have:

$$\begin{aligned} f(x) = x &\implies \frac{x}{2} + \frac{1}{x} = x \\ &\implies \frac{x^2+2}{2x} = x \\ &\implies x^2 = 2 \\ &\implies \begin{cases} x = \sqrt{2} \notin \mathbb{Q}, \\ x = -\sqrt{2} \notin \mathbb{Q}. \end{cases} \end{aligned}$$

Therefore, f admits no fixed point in $\mathbb{X} \subset \mathbb{Q}$.

3. The previous result does not contradict the fixed-point theorem because $\mathbb{X}$ is not complete, even though f is a contraction.

Exercise 9:

1. For all $x \geqslant 1$ and $t \geqslant 0$, we have:

$$\sqrt{x+t} - \sqrt{x} = \frac{(\sqrt{x+t} - \sqrt{x})(\sqrt{x+t} + \sqrt{x})}{\sqrt{x+t} + \sqrt{x}} = \frac{t}{\sqrt{x+t} + \sqrt{x}} \leqslant \frac{t}{2}.$$

2. Suppose $f(x) = \sqrt{x}$. Setting $y = x + t$, we obtain:

$$\begin{aligned} d_u(f(y), f(x)) &= |f(y) - f(x)| \\ &= \left|\sqrt{x+t} - \sqrt{x}\right| \\ &\leqslant \frac{t}{2} = \frac{1}{2}|x - y| = \frac{1}{2} d_u(x, y). \end{aligned}$$

Thus, f is a contraction on the complete space $[1, +\infty)$ (since it is closed in $\mathbb{R}$).

3. We have:

$$\begin{aligned} f(x) = x &\implies \sqrt{x} = x \\ &\implies x = x^2 \\ &\implies x(x-1) = 0 \\ &\implies \begin{cases} x = 0 \notin [1, +\infty), \\ x = 1 \in [1, +\infty). \end{cases} \end{aligned}$$

Therefore, 1 is the unique fixed point of $[1, +\infty)$.