وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 5: Topological Spaces

Exercise 1: Let $\mathbb{X} = \{1,2,3,4\}$, $\mathcal{T} = \{\emptyset, \mathbb{X}, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$ and $A = \{1,3\}, B = \{2,4\} \subset \mathbb{X}$.

1. Show that $(\mathbb{X}, \mathcal{T})$ is a topological space.
2. Find $\mathcal{N}(1)$, $\mathcal{N}(2)$, $\mathcal{N}(3)$, $\mathcal{N}(4)$.
3. Find $Cl(A)$, $Cl(B)$, $Int(A)$, $Int(B)$, A' and B'.
4. Find $\partial(A)$, $\partial(B)$, $Ext(A)$, $Ext(B)$.
5. Find $Is(A)$ et $Is(B)$.
6. Determine $\mathcal{T}_A$ et $\mathcal{T}_B$.
7. Is $(\mathbb{X}, \mathcal{T})$ a Hausdorff (separated) topological space?

Exercise 2: Let $\mathbb{X}$ an infinite set and A a subset of $\mathbb{X}$. Define the following family

$$\mathcal{T}_{Cof} = \{O \subset \mathbb{X} : \complement_{\mathbb{X}} O \text{ is finite}\} \cup \{\emptyset\}$$

1. Show that the family $\mathcal{T}_{Cof}$ is a topology on $\mathbb{X}$.
2. Determine the closed sets in $\mathbb{X}$.
3. If A is finite, find $Cl(A)$, $Int(A)$, and $\partial(A)$.
4. If A is infinite, find $Cl(A)$, $Int(A)$, and $\partial(A)$.

Exercise 3: Let f be a function from a non-empty set $\mathbb{X}$ to a topological space $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$, and let

$$\mathcal{T} = \{f^{-1}(G) : G \in \mathcal{T}_{\mathbb{Y}}\}.$$

Show that $\mathcal{T}$ is a topology on $\mathbb{X}$.

Exercise 4: Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A, B two subsets of $\mathbb{X}$. Prove that:

1. $A \subseteq B$ and A is open $\Longrightarrow A \subseteq Int(B)$.
2. $A \subseteq B \Longrightarrow Int(A) \subseteq Int(B)$.
3. $Int(A) = \mathrm{Int}(Int(A))$.
4. $Int(A \cap B) = Int(A) \cap Int(B)$.
5. $Int(A) \cup Int(B) \subset \mathrm{Int}(A \cup B)$.
6. $A \in \mathcal{N}(B) \Longleftrightarrow B \subset Int(A)$.
7. $Int(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Cl(A)$.
8. $Cl(\complement_{\mathbb{X}} A) = \complement_{\mathbb{X}} Int(A)$.
9. $\partial(A)$ is a closed set.
10. A is both open and closed $\Longleftrightarrow \partial(A) = \emptyset$.
11. A is open $\Longleftrightarrow \partial(A) \cap A = \emptyset$.
12. A is closed $\Longleftrightarrow \partial(A) \subseteq A$.

Exercise 5: Let $\mathfrak{B}$ and $\mathfrak{B}'$ be two bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Prove that $\mathcal{T} \subset \mathcal{T}'$ if and only if for every $B \in \mathfrak{B}$ and $x \in B$, there exists $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$.

Exercise 6: Prove that the function $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ is continuous in each of the following cases:

1. $(\mathbb{X}, \mathcal{T}_{\mathbb{X}}) = (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ and $f(x) = x$.
2. f is constant.
3. $\mathcal{T}_{\mathbb{X}} = \mathcal{T}_{\mathrm{Disc}}$.
4. $\mathcal{T}_{\mathbb{Y}} = \mathcal{T}_{\mathrm{Ind}}$.

Exercise 7: Prove that if $f, g : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ are continuous functions and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ is **Hausdorff**, then the following hold:

1. $A = \{x \in \mathbb{X} : f(x) = g(x)\}$ is closed in $\mathbb{X}$.
2. $\Gamma_f = \{(x, f(x)) : x \in \mathbb{X}\}$ is closed in $\mathbb{X} \times \mathbb{Y}$.

Exercise 8: Prove that every homeomorphism is an open and closed map.