وزارة التعليـم العالـي والبحـث العلمـي

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Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solution of Series 5 (Topological Spaces)

Indication 1: $Int(A)$ **is the largest open set contained in** A**.**

Indication 2: $Cl(A)$ **is the smallest closed set containing** A**.**

Exercise 1: Let $\mathbb{X} = \{1,2,3,4\}$, $\mathcal{T} = \{\emptyset, \mathbb{X}, \{3\}, \{4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\}\}$ and $A = \{1,3\}, B = \{2,4\} \subset \mathbb{X}$.

1. (c_1) We have $\mathbb{X}, \emptyset \in \mathcal{T}$ by definition.

   (c_2) It is clear that that any union of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

   (c_3) It is clear that any finite intersection of elements of $\mathcal{T}$ is an element of $\mathcal{T}$

   From (c_1), (c_2) and (c_3) we conclude that $(\mathbb{X}, \mathcal{T})$ is a topological space.

2. - $\mathcal{N}(1) = \{\mathbb{X}, \{1,3,4\}\}$.
   - $\mathcal{N}(2) = \{\mathbb{X}, \{2,3,4\}\}$.
   - $\mathcal{N}(3) = \{\mathbb{X}, \{3\}, \{1,3\}, \{2,3\}, \{3,4\}, \{1,3,4\}, \{1,2,3\}, \{2,3,4\}\}$.
   - $\mathcal{N}(4) = \{\mathbb{X}, \{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}\}$.

3. - $Cl(A) = Cl(\{1,3\}) = \{1,2,3\}$.
   - $Cl(B) = Cl(\{2,4\}) = \{1,2,4\}$.
   - $Int(A) = Int(\{1,3\}) = \{3\}$.
   - $Int(B) = Int(\{2,4\}) = \{4\}$.
   - $A' = \{1,3\}' = \{1,2\}$.
   - $B' = \{2,4\}' = \{1,2\}$.

4. • $\partial(A) = Cl(A) \setminus Int(A) = \{1,2,3\} \setminus \{3\} = \{1,2\}$.
   • $\partial(B) = Cl(B) \setminus Int(B) = \{1,2,4\} \setminus \{4\} = \{1,2\}$.
   • $Ext(A) = \complement_{\mathbb{X}} Cl(A) = \{4\}$.
   • $Ext(B) = \complement_{\mathbb{X}} Cl(B) = \{3\}$.

5. • $Is(A) = \{3\}$ and • $Is(B) = \{4\}$.

6. • $\mathcal{T}_A = \{A \cap O : O \in \mathcal{T}\} = \{\emptyset, A, \{3\}\}$.
   • $\mathcal{T}_B = \{B \cap O : O \in \mathcal{T}\} = \{\emptyset, B, \{4\}\}$.

7. For all $N \in \mathcal{N}(1)$ and $W \in \mathcal{N}(2)$ we have $N \cap W \neq \emptyset$. Then, $(\mathbb{X}, \mathcal{T})$ is not a Hausdorff (separated) topological space.

Exercise 2:

$$\mathcal{T}_{Cof} = \{O \subset \mathbb{X} : \complement_{\mathbb{X}} O \text{ is finite}\} \cup \{\emptyset\}.$$

1. Let us show that the family $\mathcal{T}_{Cof}$ is a topology on $\mathbb{X}$.

   (c_1) By definition, $\emptyset \in \mathcal{T}_{Cof}$. Moreover, since $\complement_{\mathbb{X}} \mathbb{X} = \emptyset$ (a finite set), we conclude that $\mathbb{X} \in \mathcal{T}_{Cof}$.

   (c_2) Let $\{O_i : i \in I\}$ be a family of subsets in $\mathcal{T}_{Cof}$. Then, for each $i \in I$, $\complement_{\mathbb{X}} O_i$ is a finite set. Thus,
   $$\complement_{\mathbb{X}} \left(\bigcup_{i \in I} O_i \right) = \bigcap_{i \in I} (\complement_{\mathbb{X}} O_i),$$
   is finite, as it is the intersection of finite sets. Therefore, $\bigcup_{i \in I} O_i \in \mathcal{T}_{Cof}$.

   (c_3) Let $\{O_i : i = 1, 2, \ldots, n\}$ be a finite family of subsets in $\mathcal{T}_{Cof}$. Then, for each $i = 1, 2, \ldots, n$, $\complement_{\mathbb{X}} O_i$ is finite. Thus,
   $$\complement_{\mathbb{X}} \left(\bigcap_{i=1}^{n} O_i \right) = \bigcup_{i=1}^{n} (\complement_{\mathbb{X}} O_i)$$
   is finite, as it is the finite union of finite sets. Therefore, $\bigcap_{i=1}^{n} O_i \in \mathcal{T}_{Cof}$.

   From (c_1), (c_2) and (c_3) we conclude that $(\mathbb{X}, \mathcal{T}_{Cof})$ is a topological space.

2. The closed sets of $\mathbb{X}$ are the complements of the open sets. Therefore, the closed sets in $\mathcal{T}_{Cof}$ are the finite subsets of $\mathbb{X}$ or $\mathbb{X}$ itself.

3. If A is finite.

   - $Cl(A) = A$ because A is closed.
   - $Int(A) = \emptyset$ Because it is the only open set contained in A (the other open sets are infinite).
   - $\partial(A) = Cl(A) \setminus Int(A) = A \setminus \emptyset = A$.

4. If A is infinite.

   - $Cl(A) = X$ Because it is the smallest closed set containing A.
   - $Int(A) = A$ if $\complement_{\mathbb{X}} A$ is finite and $Int(A) = \emptyset$ if $\complement_{\mathbb{X}} A$ is infinite(Indeed, if $Int(A) \neq \emptyset$, then there exists some $x \in Int(A)$, which implies the existence of an open set $O_x \in \mathcal{T}_{Cof}$ such that $x \in O_x \subset A$. Consequently, we would have $\complement_{\mathbb{X}} A \subset \complement_{\mathbb{X}} O_x$, contradicting the fact that $\complement_{\mathbb{X}} A$ is infinite (since $\complement_{\mathbb{X}} O_x$ is finite by the definition)
   - $\partial(A) = Cl(A) \setminus Int(A) = \mathbb{X} \setminus A$ if $\complement_{\mathbb{X}} A$ is finite.
   - $\partial(A) = Cl(A) \setminus Int(A) = \mathbb{X} \setminus \emptyset = \mathbb{X}$ if $\complement_{\mathbb{X}} A$ is infinite.

Exercise 3:

$$\mathcal{T} = \{f^{-1}(G) : G \in \mathcal{T}_{\mathbb{Y}}\}.$$

Let us show that $\mathcal{T}$ is a topology on $\mathbb{X}$.

(c_1) Since $\emptyset, \mathbb{Y} \in \mathcal{T}_{\mathbb{Y}}$, we deduce that $\emptyset = f^{-1}(\emptyset), \quad \mathbb{X} = f^{-1}(\mathbb{Y}) \in \mathcal{T}$.

(c_2) Let $\{O_i : i \in I\}$ be a family of subsets in $\mathcal{T}$. Then, for each $i \in I$ there exists $G_i \in \mathcal{T}$ such that $O_i = f^{-1}(G_i)$. Thus,

$$\bigcup_{i \in I} O_i = \bigcup_{i \in I} f^{-1}(G_i) = f^{-1}\left(\bigcup_{i \in I} G_i\right).$$

Since $\mathcal{T}_{\mathbb{Y}}$ is topology on $\mathbb{Y}$, we obtain $\bigcup_{i \in I} G_i \in \mathcal{T}_{\mathbb{Y}}$. Therefore, $\bigcup_{i \in I} O_i \in \mathcal{T}$.

(c_3) Let $\{O_i : i = 1, 2, \ldots, n\}$ be a family of subsets in $\mathcal{T}$. Then, for each $i = 1, 2, \ldots, n$ there exists $G_i \in \mathcal{T}$ such that $O_i = f^{-1}(G_i)$. Thus,

$$\bigcap_{i=1}^{n} O_i = \bigcap_{i=1}^{n} f^{-1}(G_i) = f^{-1}\left(\bigcap_{i=1}^{n} G_i\right).$$

Since $\mathcal{T}_{\mathbb{Y}}$ is topology on $\mathbb{Y}$, we obtain $\bigcap_{i=1}^{n} G_i \in \mathcal{T}_{\mathbb{Y}}$. Therefore, $\bigcap_{i=1}^{n} O_i \in \mathcal{T}$.

From (c_1), (c_2) and (c_3) we conclude that $(\mathbb{X}, \mathcal{T})$ is a topological space.

Exercise 4: Let $(\mathbb{X}, \mathcal{T})$ be a topological space and A, B two subsets of $\mathbb{X}$. Prove that:

1. Let A be an open subset of $\mathbb{X}$ with $A \subseteq B$. If $x \in A$, then $A \in \mathcal{N}(x)$. Since $A \subset B$, it follows that $B \in \mathcal{N}(x)$ which implies $x \in Int(B)$. Therefore, $A \subseteq Int(B)$.

2. Let $A \subseteq B$. If $x \in Int(A)$, then $A \in \mathcal{N}(x)$. Since $A \subseteq B$, it follows that $B \in \mathcal{N}(x)$ which implies $x \in Int(B)$. Therefore, $int(A) \subseteq Int(B)$.

3. On one hand, if $x \in Int(A)$, then $Int(A) \in \mathcal{N}(x)$ which implies that $x \in Int(Int(A))$. Therefore,

$$Int(A) \subseteq Int(Int(A)). \tag{i}$$

On the other hand

$$Int(Int(A)) \subseteq Int(A). \text{ (by definition)}. \tag{ii}$$

From (i) and (ii) we conclude that $Int(A) = Int(Int(A))$.

4. On one hand, we have

$$\begin{cases} A \cap B \subseteq A \\ A \cap B \subseteq B \end{cases} \Rightarrow \begin{cases} Int(A \cap B) \subseteq Int(A) \\ Int(A \cap B) \subseteq Int(B) \end{cases} \Rightarrow Int(A \cap B) \subseteq Int(A) \cap Int(B). \tag{i}$$

On the other hand we have

$$\begin{cases} Int(A) \cap Int(B) \subseteq Int(A) \subseteq A \\ Int(A) \cap Int(B) \subseteq Int(B) \subseteq B \end{cases} \Rightarrow Int(A) \cap Int()B) \subseteq A \cap B,$$

from which we conclude that

$$Int(A) \cap Int()B) \subseteq Int(A \cap B) \tag{ii}$$

From (i) and (ii) we conclude that $Int(A \cap B) = Int(A) \cap Int(B)$.

5. We have,

$$\begin{cases} Int(A) \subseteq A \\ Int(B) \subseteq B \end{cases} \Rightarrow Int(A) \cup Int()B) \subseteq A \cup B \Rightarrow Int(A) \cup Int(B) \subseteq Int(A \cup B).$$

.

6. We have,

$$\begin{aligned} A \in \mathcal{N}(B) &\iff \exists O \in \mathcal{T},\ B \subseteq O \subseteq A \\ &\iff B \subseteq Int(A). \end{aligned}$$

7. We have,

$$
\begin{aligned}
x \in Int(\complement_{\mathbb{X}} A) &\iff \complement_{\mathbb{X}} A \in \mathcal{N}(x) \\
&\iff \exists O_x \in \mathcal{T},\ x \in O_x \subset \complement_{\mathbb{X}} A \\
&\iff \exists O_x \in \mathcal{T},\ O_x \cap A = \emptyset \\
&\iff x \notin Cl(A) \\
&\iff x \in \complement_{\mathbb{X}} Cl(A)
\end{aligned}
$$

8. According to (7), we have $Int\left(\complement_{\mathbb{X}}\complement_{\mathbb{X}} A\right) = \complement_{\mathbb{X}} Cl\left(\complement_{\mathbb{X}} A\right)$. Then, $Int(A) = \complement_{\mathbb{X}} Cl\left(\complement_{\mathbb{X}} A\right)$ which shows that $\complement_{\mathbb{X}} Int(A) = Cl\left(\complement_{\mathbb{X}} A\right)$.

9. We have, $\partial(A) = Cl(A) \cap Cl\left(\complement_{\mathbb{X}} A\right)$. Then, $\partial(A)$ is closed because it is the intersection of two closed sets.

10. $\Longrightarrow$) A is clopen set $\Longrightarrow \partial(A) = Cl(A) \smallsetminus Int(A) = A \smallsetminus A = \emptyset$.

$\Longleftarrow$)

$$
\begin{aligned}
\partial(A) = \emptyset &\Longrightarrow \partial(A) = Cl(A) \smallsetminus Int(A) = \emptyset \\
&\Longrightarrow Cl(A) = Int(A) \\
&\Longrightarrow A \text{ is a clopen set.}
\end{aligned}
$$

11. $\Longrightarrow$)

$$
\begin{aligned}
A \text{ is open} &\Longrightarrow \partial(A) = Cl(A) \smallsetminus Int(A) = Cl(A) \smallsetminus A \\
&\Longrightarrow \partial(A) \cap A = \emptyset.
\end{aligned}
$$

$\Longleftarrow$)

$$
\begin{aligned}
\partial(A) \cap A = \emptyset &\Longrightarrow \left(Cl(A) \cap \complement_{\mathbb{X}}(IntA)\right) \cap A = \emptyset \\
&\Longrightarrow \complement_{\mathbb{X}}(IntA) \cap A = \emptyset. \quad (\text{because } Cl(A) \cap A \neq \emptyset) \\
&\Longrightarrow A \subseteq \complement_{\mathbb{X}}\complement_{\mathbb{X}} IntA \\
&\Longrightarrow A \subseteq IntA \\
&\Longrightarrow A = Int(A) \\
&\Longrightarrow A \text{ is an open set.}
\end{aligned}
$$

12. $\Longrightarrow$)

$$
\begin{aligned}
A \text{ is closed} &\Longrightarrow \partial(A) = Cl(A) \smallsetminus Int(A) = A \smallsetminus Int(A) \\
&\Longrightarrow \partial A \subseteq A.
\end{aligned}
$$

$\Longleftarrow$)

$$\begin{aligned}\partial A \subseteq A &\Longrightarrow Cl(A) \setminus Int(A) \subseteq A\\ &\Longrightarrow Cl(A) \subseteq A \quad (\text{because } Int(A) \subseteq A)\\ &\Longrightarrow Cl(A) = A\\ &\Longrightarrow Cl(A) \text{ is a closed set.}\end{aligned}$$

Exercise 5:

$\Longrightarrow$) Suppose that $\mathcal{T} \subset \mathcal{T}'$. Let $x \in B \in \mathfrak{B} \subset \mathcal{T}$. Then $x \in B \in \mathcal{T}'$ because $\mathcal{T} \subset \mathcal{T}'$, hence $x \in B = \bigcup_{B' \in \mathfrak{B}'} B'$. Therefore, there exists $B' \in \mathfrak{B}'$ such that $B' \subset B$.

$\Longleftarrow$) Let $O \in \mathcal{T}$. On the one hand, for all $x \in O$, there exists $B \in \mathfrak{B}$ such that $x \in B \subset O$. But by hypothesis, there exists $B' \in \mathfrak{B}'$ such that $x \in B' \subset B$. Hence, for all $x \in O$, there exists $B' \in \mathfrak{B}'$ such that $x \in B' \subset O$, which implies

$$\bigcup_{B' \in \mathfrak{B}'} B' \subseteq O. \tag{i}$$

On the other hand,

$$O = \bigcup_{x \in O} \{x\} \subseteq \bigcup_{B' \in \mathfrak{B}'} B'. \tag{ii}$$

Finally, from (i) and (ii), we conclude that $O = \bigcup_{B' \in \mathfrak{B}'} B'$, and therefore $O \in \mathcal{T}'$.

Exercise 6:

1. Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}}) = (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ and $f(x) = x$. Then, for any open set $O \in \mathcal{T}_{\mathbb{Y}} = \mathcal{T}_{\mathbb{X}}$, we have $f^{-1}(O) = O \in \mathcal{T}_{\mathbb{X}}$, since $f^{-1}(O) = O$ by definition of f. Therefore, the preimage of an open set is an open set, implying that f is continuous.

2. Let f be a constant function. Then, for each $x \in \mathbb{X}$, we have $f(x) = k \in \mathbb{Y}$. This implies that
$$\begin{cases} f^{-1}(O) = \emptyset, & \text{if } k \notin O,\\ f^{-1}(O) = \mathbb{X}, & \text{if } k \in O,\end{cases}$$
for each $O \in \mathcal{T}_{\mathbb{Y}}$. Therefore, $f^{-1}(O) \in \mathcal{T}_{\mathbb{X}}$, implying that f is continuous.

3. Let $\mathcal{T}_{\mathbb{X}} = \mathcal{T}_{\text{Disc}}$. Then, $f^{-1}(O) \in \mathcal{T}_{\mathbb{X}}$ for each $O \in \mathcal{T}_{\mathbb{Y}}$, since every subset in the discrete topology is open. Thus, f is continuous.

4. Let $\mathcal{T}_{\mathbb{Y}} = \mathcal{T}_{\text{Ind}} = \{\emptyset, \mathbb{Y}\}$. Then we have $f^{-1}(\emptyset) = \emptyset \in \mathcal{T}_{\mathbb{X}}$ and $f^{-1}(\mathbb{Y}) = \mathbb{X} \in \mathcal{T}_{\mathbb{X}}$. Therefore, f is continuous.

Exercise 7:

1. It suffices to show that $\complement_{\mathbb{X}} A$ is open in $\mathbb{X}$.

   Let $b \in \complement_{\mathbb{X}} A$. Then $f(b) \neq g(b)$, and since $\mathbb{Y}$ is Hausdorff, there exist two open sets $O_1, O_2 \in \mathcal{T}_{\mathbb{Y}}$ such that $f(b) \in O_1$, $g(b) \in O_2$, and $O_1 \cap O_2 = \emptyset$. Thus, $b \in f^{-1}(O_1) \cap g^{-1}(O_2) \in \mathcal{T}_{\mathbb{X}}$ because f and g are continuous and $O_1, O_2 \in \mathcal{T}_{\mathbb{Y}}$. Furthermore, $f^{-1}(O_1) \cap g^{-1}(O_2) \subset \complement_{\mathbb{X}} A$, which shows that $\complement_{\mathbb{X}} A$ is open. Therefore, A is closed.

2. It suffices to show that $\complement_{\mathbb{X}\times\mathbb{Y}} \Gamma_f$ is open in $\mathbb{X} \times \mathbb{Y}$.

   Let $(x, y) \in \complement_{\mathbb{X}\times\mathbb{Y}} \Gamma_f$. Then $(x, y) \notin \Gamma_f$, and thus $y \neq f(x)$. Since $\mathbb{Y}$ is Hausdorff, there exist two open sets $O_1, O_2 \in \mathcal{T}_{\mathbb{Y}}$ such that $f(x) \in O_1$, $y \in O_2$, and $O_1 \cap O_2 = \emptyset$. Since f is continuous, $x \in O_3 = f^{-1}(O_1) \in \mathcal{T}_{\mathbb{X}}$. Hence, $(x, y) \in O_3 \times O_2 \subseteq \complement_{\mathbb{X}\times\mathbb{Y}} \Gamma_f$, which shows that $\complement_{\mathbb{X}\times\mathbb{Y}} \Gamma_f$ is open. Therefore, Γ_f is closed.

Exercise 8: Let $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ an homeomorphism

1. **Is f open?**

   Let $O \in \mathcal{T}_{\mathbb{X}}$ and $g = f^{-1}$. Then $g^{-1}(O) = f(O)$, which is open since $g = f^{-1}$ is continuous. Therefore, f is an open map.

2. **Is f closed?**

   Let $F \in \mathcal{T}_{\mathbb{X}}$ and $g = f^{-1}$. Then $g^{-1}(F) = f(F)$, which is closed since $g = f^{-1}$ is continuous. Therefore, f is a closed map.