وزارة التعليــم العالــي والبحــث العلمــي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Final Exam in Topology for Second-Year LMD Mathematics Students (2024/2025)

Group:	Last Name:	First Name:

Exercise 1 (7.5 pts): Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ be a topological space and $A \subset \mathbb{X}$. Complete the following expressions:

1. Any member of $\mathcal{T}_{\mathbb{X}}$ is called ....
2. The topology induced by Euclidean metric on $\mathbb{R}$ is called ....
3. The topological space in which all the subsets of $\mathbb{X}$ are clopen is called ....
4. If $Int\left(\complement_{\mathbb{X}} A\right) = \complement_{\mathbb{X}} A$, then A is ....(open/closed).
5. The topology in which every finite sets are closed is called ....
6. If $Cl(A) = \mathbb{X}$, then A is ....
7. If A is a neighborhood of each of its points, then A is ....
8. The largest open set contained in A is called ....
9. The set which is the intersection of $Cl(A)$ and $Cl\left(\complement_{\mathbb{X}} A\right)$ is called ....
10. The intersection of all closed sets containing A is called...

Exercise 2 (8.5 pts):

1. Recall the definition of a metric space.
2. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function defined by:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

   (a) Prove that d is indeed a metric.
   (b) Provide an example of an open ball of radius 1 centered at $(0, 0)$ for this metric.

3. Prove that every metric space is a topological space.

Exercise 3 (4pts):

1. Recall the definition of a complete metric space.
2. Show that every complete subset in a metric space $(\mathbb{X}, d_{\mathbb{X}})$ is closed.

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جامعة سطيف 1 - فرحات عباس

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Exercise 1 (7.5 pts): Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ be a topological space and $A \subset \mathbb{X}$. Complete the following expressions:

1. Any member of $\mathcal{T}_{\mathbb{X}}$ is called ....open ball set.. 0,75
2. The topology induced by Euclidean metric on $\mathbb{R}$ is called ....usual topology......... 0,75
3. The topological space in which all the subsets of $\mathbb{X}$ are clopen is called discrete topological space. 0,75
4. If $Int\left(\complement_{\mathbb{X}} A\right) = \complement_{\mathbb{X}} A$, then A is ....closed.. 0,75
5. The topology in which every finite sets are closed is called ....cofinite topology....... 0,75
6. If $Cl(A) = \mathbb{X}$, then A is ....dense in $\mathbb{X}$.. 0,75
7. If A is a neighborhood of each of its points, then A is ....open.................... 0,75
8. The largest open set contained in A is called ....interior of A............ 0,75
9. The set which is the intersection of $Cl(A)$ and $Cl\left(\complement_{\mathbb{X}} A\right)$ is called ....boundary of A...... 0,75
10. The intersection of all closed sets containing A is called...............closure of A......... 0,75

Exercise 2 (8.5 pts):

1. **Recall the definition of a metric space:**

Definition 1. *A metric space is a pair* $(\mathbb{X}, d)$*, where* $\mathbb{X}$ *is a set and* $d: \mathbb{X}\times\mathbb{X} \to \mathbb{R}$ *is a function, called a metric, that satisfies the following properties for all* $x, y, z \in \mathbb{X}$*:*

(a) ***Non-negativity:*** $d(x,y) \geq 0$, *and* $d(x,y) = 0 \iff x = y$. 0,5

(b) ***Symmetry:*** $d(x,y) = d(y,x)$. 0,5

(c) ***Triangle inequality:*** $d(x,z) \leq d(x,y) + d(y,z)$. 0,5

2. **Let** $d: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ **be a function defined by:**

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

(a) **Prove that** d **is a metric on** $\mathbb{R}^2$**:**
We need to check the three properties of a metric:

i. **Non-negativity:**

$$|x_1 - x_2| \geq 0 \quad \text{and} \quad |y_1 - y_2| \geq 0 \quad \Rightarrow \quad d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2| \geq 0.$$

Moreover, $d((x_1,y_1),(x_2,y_2)) = 0 \iff |x_1 - x_2| = 0$ and 0,5 $|y_1 - y_2| = 0 \Rightarrow x_1 = x_2$ and $y_1 = y_2$. 0,5

ii. **Symmetry:**

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1| = d((x_2,y_2),(x_1,y_1)).$$

0,5 0,5

iii. **Triangle inequality:**
Let $z = (x_3, y_3)$. We have:

$$d((x_1,y_1),(x_3,y_3)) = |x_1 - x_3| + |y_1 - y_3|. \quad 0,5$$

Using the triangle inequality for absolute values:

$$0,5 \quad |x_1 - x_3| \leq |x_1 - x_2| + |x_2 - x_3| \quad \text{and} \quad |y_1 - y_3| \leq |y_1 - y_2| + |y_2 - y_3|. \quad 0,5$$

Adding these inequalities gives:

$$d((x_1,y_1),(x_3,y_3)) \leq d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),(x_3,y_3)). \quad 0,5$$

Therefore, d satisfies all the properties of a metric.

(b) **Give an example of an open ball of radius** 1 **centered at** $(0,0)$ **for this metric:**
An open ball of radius 1 centered at $(0,0)$ is defined as:

$$B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : d((0,0),(x,y)) < 1\}. \quad 0,5$$

Using the definition of $d((0,0),(x,y)) = |x| + |y|$, the open ball becomes:

$$B((0,0),1) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\}. \quad 0,5$$

This represents a diamond-shaped region in the plane, bounded by the lines:

$$x + y = 1, \quad x - y = 1, \quad -x + y = 1, \quad -x - y = 1.$$

3. **Prove that every metric space is a topological space:**
Given a metric space $(\mathbb{X}, d)$, we can define a topology on $\mathbb{X}$ as follows: - A subset $U \subseteq \mathbb{X}$ is *open* if for every $x \in U$, there exists $\epsilon > 0$ such that the open ball $B(x, \epsilon) = \{y \in \mathbb{X} : d(x, y) < \epsilon\}$ is contained in U.

Proof:

(a) **The empty set and $\mathbb{X}$ are open:**
The empty set contains no points, so the condition is satisfied vacuously. 0,5 For $\mathbb{X}$, every point $x \in \mathbb{X}$ has $B(x, \epsilon) \subseteq \mathbb{X}$. 0,5

(b) **Arbitrary unions of open sets are open:**
Let $\{U_\alpha\}_{\alpha \in I}$ be a family of open sets. If $x \in \bigcup_{\alpha \in I} U_\alpha$, then $x \in U_{\alpha_0}$ for some $\alpha_0 \in I$. Since U_{α_0} is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U_{\alpha_0} \subseteq \bigcup_{\alpha \in I} U_\alpha$. 0,5

(c) **Finite intersections of open sets are open:**
Let $U_1, U_2, \ldots, U_n$ be open sets. If $x \in \bigcap_{i=1}^{n} U_i$, then for each i, there exists $\epsilon_i > 0$ such that $B(x, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$. Then $B(x, \epsilon) \subseteq \bigcap_{i=1}^{n} U_i$. 0,5

Thus, the topology defined by the metric satisfies the axioms of a topological space. Hence, every metric space is a topological space.

Exercise 3 (4pts):

1. **Recall the definition of a complete metric space.**

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> **Definition 2.** *A metric space $(\mathbb{X}, d)$ is said to be complete if every Cauchy sequence in $(\mathbb{X}, d)$ converges to a limit that is also in $\mathbb{X}$.*

2. **Show that every complete subset in a metric space $(\mathbb{X}, d_{\mathbb{X}})$ is closed.**

0,5

Let A be a complete subset of $\mathbb{X}$, and let $x \in Cl(A)$. Then there exists a sequence (x_n) of elements in A such that $x_n \longrightarrow x$ (see Proposition (2.4)). 0,5 Since (x_n) is a Cauchy sequence in A, and A is complete, it follows that $x \in A$. This shows that A is closed. 0,5

0,5