وزارة التعليم العالي والبحث العلمي

Sétif 1 University-Ferhat ABBAS
Faculty of Sciences
Department of Mathematics

Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 6: Compact Spaces

Exercise 1: Using (only) the definition of compactness, show that $\mathbb{R}$, $[0,+\infty[$, and $]0,1[$ are not compact in $(\mathbb{R},|\cdot|)$.

Exercise 2: Determine whether the set A is compact in $\mathbb{X}$ in the following cases:

1) $A=\mathbb{Q}$, $\mathbb{X}=\mathbb{R}$,

2) $A=\left\{\frac{1}{n} : n\in\mathbb{N}^*\right\}$, $\mathbb{X}=\mathbb{R}$,

3) A is an infinite set in $(\mathbb{X},\delta)$,

4) $A=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$, $\mathbb{X}=\mathbb{R}^2$,

5) $A=\left\{(x,y)\in\mathbb{R}^2 : x\geq 1,\ 0\leq y\leq \frac{1}{x}\right\}$, $\mathbb{X}=\mathbb{R}^2$,

6) $A=\left\{\left(x,\sin\frac{1}{x}\right)\in\mathbb{R}^2 : 0<x\leq 1\right\}$, $\mathbb{X}=\mathbb{R}^2$.

Exercise 3: Consider the metric space $(\mathbb{Q},d)$, where $d(x,y)=|x-y|$, and the set $A=\{x\in\mathbb{Q} : 2<x^2<3\}$. Show that A is closed and bounded but not compact.

Exercise 4: Let $(\mathbb{X},\mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ be two topological spaces, where $(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ is separated, and let $f:(\mathbb{X},\mathcal{T}_{\mathbb{X}})\longrightarrow(\mathbb{Y},\mathcal{T}_{\mathbb{Y}})$ be a continuous map. Show that if B is a compact subset of $\mathbb{Y}$, then $f^{-1}(B)$ is closed. Find an example that shows that $f^{-1}(B)$ is not necessarily compact.

Exercise 5: Let $(\mathbb{X},\mathcal{T})$ be a separated topological space. Prove the following:

1. The finite union of compact sets in $\mathbb{X}$ is compact.
2. The arbitrary union of compact sets in $\mathbb{X}$ is not necessarily compact.

Exercise 6: Show that the discrete topological space $(\mathbb{X},\mathcal{T}_{disc})$ is compact if and only if $\mathbb{X}$ is finite.

Exercise 7: Let $(\mathbb{X},d)$ be a metric space and $A\subset\mathbb{X}$. Prove the following:

1. If A is precompact, then $Cl(A)$ is also precompact.
2. If A is precompact, then A is bounded.

Exercise 8: Show that every finite subset of $(\mathbb{R}, |\cdot|)$ is compact.

Exercise 9: Let $(\mathbb{X}, d)$ be a metric space, and let $A, B \subset \mathbb{X}$ such that $A \cap B = \emptyset$. Prove the following:

1. If A is compact and B is closed, then $\operatorname{dist}(A, B) > 0$.
2. Is it true that $\operatorname{dist}(A, B) \neq 0$ if A and B are both closed?