وزارة التعليم العالي والبحث العلمي

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Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solution of series 6: Compact Spaces

Exercise 1: Recall that a subset $A \subset \mathbb{R}$ is compact if every open cover of A has a finite subcover, or equivalently, if every sequence in A has a convergent subsequence. In the context of metric spaces, compactness can also be characterized by the property that every sequence in A has a convergent subsequence whose limit lies in A, or by the fact that A is closed and bounded. We will use this definition to show that the sets $\mathbb{R}$, $[0, +\infty)$, and $(0, 1)$ are not compact.

1. **$\mathbb{R}$ is not compact**

   Let $\mathcal{K} = \{]-n, n[: n \in \mathbb{N}\}$, where each $(-n, n)$ is an open interval centered at 0 and has radius n.

   (a) $\mathcal{K}$ is an open cover of $\mathbb{R}$

   First, observe that the union of all sets in $\mathcal{K}$ covers $\mathbb{R}$, since for any $x \in \mathbb{R}$, there exists some $n \in \mathbb{N}$ such that $x \in (-n, n)$. Specifically, for any $x \in \mathbb{R}$, we can choose $n \geq |x|$, and it follows that $x \in (-n, n)$. Thus, $\mathcal{K}$ is indeed an open cover of $\mathbb{R}$.

   (b) No finite subcover exists

   Suppose, for the sake of contradiction, that there is a finite subcover $\mathcal{K}_0 = \{(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)\}$ of $\mathcal{K}$. Since the subcover is finite, we have a finite number of intervals. Let $N = \max\{n_1, n_2, \ldots, n_k\}$. This means that $\mathcal{K}_0$ covers the set $(-N, N)$, but the interval (N, ∞) is not covered by $\mathcal{K}_0$. For example, the point $N + 1$ is not contained in any of the sets $(-n_i, n_i)$ because $N + 1 > n_i$ for all i.

This contradiction shows that $\mathbb{R}$ cannot be covered by a finite number of sets from $\mathcal{K}$.

2. $[0,+\infty)$ **is not compact**

   The family $\mathcal{K} = \{(-1,n) : n \in \mathbb{N}\}$ is an open cover of $[0,+\infty)$ that has no finite subcover of $[0,+\infty)$, because for any finite subfamily $\mathcal{K}_0 = \{(-1,n_i) : i = 1,\ldots,p\}$ of $\mathcal{K}$, we have
$$\bigcup_{i=1}^{p}(-1,n_i) = (-1,N),$$
   where $N = \max\limits_{1\leqslant i\leqslant p} n_i$. Thus, $[0,+\infty)$ is not covered by $\mathcal{K}_0$. It follows that $[0,+\infty)$ is not compact.

3. $(0,1)$ **is not compact**

   The family $\mathcal{K} = \left\{\left(\frac{1}{n},1\right) : n \in \mathbb{N}^*\right\}$ is an open cover of $(0,1)$ that has no finite subcover of $(0,1)$, because for any finite subfamily $\mathcal{K}_0 = \left\{\left(\frac{1}{n_i},1\right) : i = 1,\ldots,p\right\}$ of $\mathcal{K}$, we have
$$\bigcup_{i=1}^{p}\left(\frac{1}{n_i},1\right) = \left(\frac{1}{N},1\right),$$
   where $N = \max\limits_{1\leqslant i\leqslant p} n_i$. Thus, $[0,+\infty)$ is not covered by $\mathcal{K}_0$. It follows that $[0,+\infty)$ is not compact.

Exercise 2:

1. $A = \mathbb{Q}$ is not compact in $\mathbb{X} = \mathbb{R}$ because it is not closed.
2. $A = \left\{\frac{1}{n} : n \in \mathbb{N}^*\right\}$ is not compact in $\mathbb{X} = \mathbb{R}$ because it is not closed.
3. The family $\mathcal{K} = \{\{x\} : x \in A\}$ is an open cover of A
$$A = \bigcup_{x\in A}\{x\}$$
   that has no finite subcover of A, since any finite subfamily $\mathcal{K}_0 = \{\{x_i\} : i = 1,\ldots,p\}$ does not cover A (as A is infinite). Therefore, A is not compact.

4. $A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is compact in $\mathbb{X} = \mathbb{R}^2$ because it is bounded and closed.

5. $A = \left\{(x,y) \in \mathbb{R}^2 : x \geqslant 1,\ 0 \leqslant y \leqslant \frac{1}{x}\right\}$ is not compact in $\mathbb{X} = \mathbb{R}^2$ because it is unbounded.

6. $A = \left\{\left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 : 0 < x \leqslant 1\right\}$ is not compact in $\mathbb{X} = \mathbb{R}^2$ because it is not closed. Indeed, the sequence $\left(\dfrac{1}{n\pi}, 0\right)$ belongs to A and converges to $(0,0)$, but $(0,0) \notin A$.

Exercise 3: Consider the metric space $(\mathbb{Q}, d)$, where $d(x,y) = |x-y|$, and the set

$$A = \left\{x \in \mathbb{Q} : 2 < x^2 < 3\right\}.$$

1. It is clear that A is bounded.

2. We have

$$A = \left\{x \in \mathbb{Q} : -\sqrt{3} < x < -\sqrt{2}\right\} \cup \left\{x \in \mathbb{Q} : \sqrt{2} < x < \sqrt{3}\right\}.$$

   Let $A_1 = \left[-\sqrt{3}, -\sqrt{2}\right] \bigcap \mathbb{Q}$ and $A_2 = \left[\sqrt{2}, \sqrt{3}\right] \bigcap \mathbb{Q}$. Then, A_1 and A_2 are closed in $\mathbb{Q}$. Moreover, since $A = A_1 \cup A_2$, it follows that A is closed in $\mathbb{Q}$.

3. Let $G = \{G_n : n \geq 1\}$, where

$$G_n = \left\{x \in \mathbb{Q} : 2 + \frac{1}{n} < x^2 < 3 - \frac{1}{n}\right\}.$$

   Then, G is an open cover of A that has no finite subcover of A. Therefore, A is not compact.

Exercise 4: Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, where $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ is separated, and let $f : (\mathbb{X}, \mathcal{T}_{\mathbb{X}}) \longrightarrow (\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be a continuous map.

Since B is compact in $\mathbb{Y}$, which is a Hausdorff space, it follows that B is closed. Consequently, $f^{-1}(B)$ is closed because f is a continuous function.

Counterexample: Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 0, \quad \forall x \in \mathbb{R}.$$

Let $B = \{0\}$. Then, B is compact, but

$$f^{-1}(B) = \mathbb{R},$$

is not compact.

Exercise 5: Let $(\mathbb{X}, \mathcal{T})$ be a separated topological space.

1. It is sufficient to show that the union of two compact sets is compact. Let K_1 and K_2 be two compact sets. Define $K = K_1 \cup K_2$, and let $\{O_i : i \in I\}$ be an open cover of K. Then, $\{O_i : i \in I\}$ is also an open cover of K_1 and K_2.

   Since K_1 and K_2 are compact, there exist finite subsets $I_1, I_2 \subset I$ such that

$$K_1 \subseteq \bigcup_{i \in I_1} O_i \quad \text{and} \quad K_2 \subseteq \bigcup_{i \in I_2} O_i.$$

   Therefore,

$$K = K_1 \cup K_2 \subseteq \bigcup_{i \in I_1 \cup I_2} O_i.$$

   Since $I_1 \cup I_2$ is finite, this shows that K is compact.

2. Consider the family $\{K_n = [1, n+1] : n \geq 1\}$. It is clear that each K_n is compact for all $n \geq 1$, but the union

$$\bigcup_{n \geq 1} K_n = [1, +\infty),$$

   is not compact.

Exercise 6:

$\Longrightarrow$) Suppose that the discrete space $(\mathbb{X}, \mathcal{T}_{\text{disc}})$ is compact. By definition, every subset of $\mathbb{X}$ is open. Consider the open cover $\mathcal{U} = \{\{x\} \mid x \in \mathbb{X}\}$, which clearly covers $\mathbb{X}$. By compactness, there must exist a finite subcover, meaning there are finitely many elements $x_1, x_2, \ldots, x_n$ such that

$$\mathbb{X} = \{x_1, x_2, \ldots, x_n\}.$$

Thus, $\mathbb{X}$ is finite.

$\Longleftarrow$) Suppose that $\mathbb{X}$ is finite. Let $\mathcal{U}$ be an arbitrary open cover of $\mathbb{X}$. Since $\mathbb{X}$ is finite, the elements of $\mathcal{U}$ already form a finite collection, and there exists a finite subcover that still covers $\mathbb{X}$. Hence, $(\mathbb{X}, \mathcal{T}_{\text{disc}})$ is compact.

Therefore, a discrete space is compact if and only if it is finite.

Exercise 7: Let $(\mathbb{X}, d)$ be a metric space and $A \subset \mathbb{X}$.

1. Suppose that A is precompact. Then, for every $r > 0$, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, r).$$

If $y \in Cl(A) \setminus A$, then $B(y, r) \cap A \neq \emptyset$. Let $z \in B(y, r) \cap A$, so there exists an index $1 \leq k \leq n$ such that $z \in B(y, r) \cap B(x_k, r)$. It follows that

$$d(y, x_k) \leq d(y, z) + d(z, x_k) < r + r = 2r.$$

Therefore, $y \in B(x_k, 2r)$, which shows that

$$Cl(A) \subseteq \bigcup_{i=1}^{n} B(x_i, 2r).$$

Hence, $Cl(A)$ is precompact.

2. If A is precompact, then for every $r > 0$, there exist $x_1, x_2, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B(x_i, r).$$

Thus, the diameter of A satisfies

$$\text{diam}(A) \leq \sum_{i=1}^{n} B(x_i, r) = 2nr,$$

which implies that A is bounded.

Exercise 8: Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of $\mathbb{R}$, and define

$$M = \max\{|a_1|, |a_2|, \ldots, |a_n|\}.$$

Then, A is bounded because for all $1 \leq i \leq n$, we have $|a_i| \leq M$.

Moreover, A is closed since it can be written as

$$A = \bigcup_{i=1}^{n} \{a_i\},$$

which is a finite union of closed sets in $\mathbb{R}$.

Since A is both bounded and closed in $\mathbb{R}$, it is compact.

Exercise 9: Let $(\mathbb{X}, d)$ be a metric space, and let $A, B \subset \mathbb{X}$ such that $A \cap B = \emptyset$.

1. Suppose that A is compact and B is closed. Then, we have $Cl(A) = A$ and $Cl(B) = B$. Consider the function $f : A \to \mathbb{R}$ defined by

$$f(x) = d(x, B), \quad \forall x \in A.$$

Since f is continuous on the compact set A, it is bounded and attains its minimum on A. Hence, there exists $a \in A$ such that

$$f(a) = \inf_{x \in A} d(x, B) = d(A, B).$$

Since $A \cap B = \emptyset$, we deduce that $d(A, B) > 0$, because if $d(A, B) = 0$, then

$$f(a) = d(a, B) = 0,$$

which implies that $a \in Cl(B) = B$, contradicting $A \cap B = \emptyset$.

2. Consider the two sets

$$A = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$$

and

$$B = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

Both A and B are closed in $\mathbb{R}^2$, but

$$d(A, B) = 0$$

since $y = \dfrac{1}{x} \to 0$ as $x \to +\infty$.