Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Series 7: Connected Spaces

Exercise 1: Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces, and let $f : \mathbb{X} \to \mathbb{Y}$ be a function. Show that if f is continuous and $\mathbb{X}$ is connected, then $f(\mathbb{X})$ is connected.

Exercise 2: Let $(\mathbb{X}, d)$ be a metric space. Show that if $\mathbb{X}$ is connected and $f : \mathbb{X} \to \mathbb{R}$ is a continuous function such that $|f(x)| = 1$ for all $x \in \mathbb{X}$, then f is constant.

Exercise 3: Show that every path-connected space is connected.

Exercise 4: Let $(\mathbb{X}, \mathcal{T})$ be a topological space. Show that if $f, g : [0, 1] \to \mathbb{X}$ are two paths from x to y and from y to z, respectively, then the function

$$h(t) = \begin{cases} f(2t), & \text{if } t \in \left[0, \frac{1}{2}\right], \\ g(2t-1), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

is a path from x to z in $\mathbb{X}$.

Exercise 5: Show that every continuous function $f : [a, b] \to [a, b]$ has a fixed point $x \in [a, b]$.

Exercise 6: Let $(\mathbb{X}, \mathcal{T}_{\mathbb{X}})$ and $(\mathbb{Y}, \mathcal{T}_{\mathbb{Y}})$ be two topological spaces such that $\mathbb{X}$ is path-connected and $f : \mathbb{X} \to \mathbb{Y}$ is continuous and surjective. Show that $\mathbb{Y}$ is path-connected.

Exercise 7: Show that $\mathbb{X} = C([a, b])$, equipped with the metric

$$d(f, g) = \sup_{t \in [a,b]} |f(t) - g(t)|$$

is path-connected, and hence connected.

Exercise 8: Suppose that A and B are connected subsets of $(\mathbb{X}, \mathcal{T})$ such that

$$Cl(A) \cap B \neq \emptyset.$$

Show that $A \cup B$ is connected.