وزارة التعليم العالي والبحث العلمي

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Introduction to Metric and Topological Spaces

Mathematics Bachelor's Degree - LMD - 3^{rd} Semester

Solution of series 7: Connected Spaces

Exercise 1: Let $f : \mathbb{X} \to \mathbb{Y}$ be a continuous function. We prove the following contrapositive statement:

$$\text{If } f(\mathbb{X}) \text{ is disconnected, then } \mathbb{X} \text{ is disconnected.}$$

Suppose that $f(\mathbb{X})$ is disconnected. Then there exists a non-empty clopen subset $G \subsetneq f(\mathbb{X})$. Since f is continuous, the non-empty preimage $f^{-1}(G)$ is clopen in $\mathbb{X}$ and $f^{-1}(G) \neq \mathbb{X}$, which shows that $\mathbb{X}$ is disconnected.

Since we have proven the contrapositive, it follows that if $\mathbb{X}$ is connected, then $f(\mathbb{X})$ must also be connected.

Exercise 2: Let $(\mathbb{X}, d)$ be a metric space. Suppose that $f : \mathbb{X} \to \mathbb{R}$ is a continuous function such that $|f(x)| = 1$ for all $x \in \mathbb{X}$. We prove the following contrapositive statement:

$$\text{If } f \text{ is not constant, then } \mathbb{X} \text{ is disconnected.}$$

If f is not constant, then its image is

$$f(\mathbb{X}) = \{-1, 1\}.$$

This implies that

$$\mathbb{X} = f^{-1}(\{-1\}) \cup f^{-1}(\{1\}).$$

Since f is continuous, the preimages $f^{-1}(\{-1\})$ and $f^{-1}(\{1\})$ are closed in $\mathbb{X}$.

Moreover, these two sets are disjoint and non-empty, forming a partition of $\mathbb{X}$ into two disjoint closed sets. This shows that $\mathbb{X}$ is not connected.

Since we have proven the contrapositive, it follows that if $\mathbb{X}$ is connected, then f must be constant.

Exercice 3: Suppose that X is path-connected and that $g : X \to \{0,1\}$ is a continuous function. If g is not constant, then there exist points $x_1, x_2 \in X$ such that $g(x_1) = 0$ and $g(x_2) = 1$. Let $f : [0,1] \to X$ be a path in X such that

$$f(0) = x_1 \quad \text{and} \quad f(1) = x_2.$$

Then the composition $g \circ f : [0,1] \to \{0,1\}$ is continuous and surjective.

However, the interval $[0,1]$ is connected, while $\{0,1\}$ is a disconnected space. This contradicts the fact that a continuous image of a connected space must also be connected.

Thus, our initial assumption that g is not constant must be false, which means that g must be constant. Then, $\mathbb{X}$ is connected.

Exercice 4: We have $h(0) = f(0) = x$, $h\left(\frac{1}{2}\right) = f(1) = y = g(0)$, and $h(1) = g(1) = z$. Moreover, since $f(2t)$ and $g(2t-1)$ are compositions of continuous functions, they are themselves continuous, which implies that h is also continuous. Then, the function h is a path from x to z in $\mathbb{X}$.

Exercise 5: Since $f([a,b]) \subset [a,b]$, it follows that $f(a) \geqslant a$ and $f(b) \leqslant b$. Define $g(x) = f(x) - x$. Then, g is continuous, and we have $g(a) \geqslant 0$ and $g(b) \leqslant 0$. Thus, by the intermediate value theorem, there exists $c \in (a,b)$ such that $g(c) = 0$, or equivalently, $f(c) = c$.

Exercise 6: Let $y_1, y_2 \in Y$. Since f is surjective, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $g : [0,1] \to X$ be a continuous path in X from x_1 to x_2. Then, the composition $f \circ g : [0,1] \to Y$ is a continuous path in Y from y_1 to y_2. Thus, Y is path-connected.

Exercise 7: Suppose that f, g are two arbitrary elements of $C([0,1])$. We define a path $h : [0,1] \to C([0,1])$ by

$$h(t) = tf + (1-t)g.$$

Then, for each $t \in [0,1]$, the function $h(t)$ is continuous, hence it belongs to $C([0,1])$.

Moreover, the function h is continuous since

$$d_\infty(h(t), h(s)) = \sup_{x\in[0,1]} |(t-s)f(x) + (s-t)g(x)| \leq |t-s|(M_f + M_g),$$

where $|f(x)| \leq M_f$ and $|g(x)| \leq M_g$ for all $x \in [0,1]$.

For any $\varepsilon > 0$, setting $\delta = \dfrac{\varepsilon}{M_f + M_g}$, we obtain

$$d_\infty(h(t), h(s)) < \varepsilon \quad \text{whenever} \quad |t - s| < \delta.$$

Finally, since $h(0) = g$ and $h(1) = f$, the function h is a continuous path in $C([0,1])$ from g to f. Thus, $C([0,1])$ is path-connected and therefore connected.

Exercice 8: Consider the continuous function $f : A \cup B \to \{0,1\}$ with the discrete metric. Since A and B are connected, the restrictions $f|_A = c_A : A \to \{0,1\}$ and $f|_B = c_B : B \to \{0,1\}$ are constant and take the value 0 or 1. Let $f|_A(x) = c_A$, $\forall x \in A$ and $f|_B(x) = c_B, \forall x \in B$.

Note that the closure $Cl(A)$ is also connected, since A is connected and $f(Cl(A)) = \{c_A\}$. Let $b \in Cl(A) \cap B$, then $f(b) = c_A = c_B$. Thus, $f(A \cup B) = c_A = c_B$, which implies that f is constant on $A \cup B$, proving that $A \cup B$ is connected.