Ferhat Abbas University-Setif 1
Faculty of Sciences
Department of Chemistry/Physics

Solved Exercises

Exercise 1. Solve the following system using the Jacobi method and determine the number of iterations required to obtain an error $\varepsilon = \|x^k - x^{k-1}\| \leq 10^{-4}$, taking the initial vector $X^0 = (0,0,0)^t$.

$$\begin{cases} 4x_1 + 1x_2 + x_3 & = 4 \\ -x_1 + 2x_2 & = 2 \\ 2x_1 + x_2 + 4x_3 & = 9 \end{cases}$$

Solution .

For each iteration k, the iterative scheme of the Jordan method is written in this case as follows :

$$\begin{cases} i=1, & x_1^{k+1} & = \frac{1}{4}\left(4 - 4x_2^k - x_3^k\right) \\ i=2, & x_2^{k+1} & = \frac{1}{2}\left(2 + x_1^k\right) \\ i=3, & x_3^{k+1} & = \frac{1}{4}\left(9 - 2x_1^k - x_2^k\right) \end{cases}$$

Starting from $X^0 = (0,0,0)^t$, to achieve the prescribed accuracy, we perform 12 iterations, the results of which are presented in the following table.

k	x_k^1	x_k^2	x_k^3
0	0	0	0
1	1	1	2.25
2	-0.0625	1.5	1.5
3	-0.125	0.9688	1.9063
4	0.0391	0.9375	2.0703
5	0.0137	1.0195	1.9961
6	-0.0088	1.0068	1.9883
7	-0.0005	0.9956	2.0027
8	0.0015	0.9998	2.0013
9	-0.0002	1.0008	1.9993
10	-0.0002	0.9999	1.9999
11	0.0001	0.9999	2.0001
12	0	1	2

Exercise 2. Consider the following system

$$\begin{cases} 2x_1 - x_2 + x_3 & = 3 \\ x_1 + 7x_2 - 3x_3 & = 6 \\ -x_1 + 3x_2 + 4x_3 & = 17 \end{cases}$$

a- Starting from $X^0 = (0,0,0)^t$, determine the first six iterations of the Jacobi and Gauss-Seidel methods.

b- Given that the exact solution is $X = (1,2,3)^t$, what can we conclude ?

Solution .

a- For each iteration k, the iterative scheme of the Jordan method is written in this case as follows :

$$\begin{cases} i=1, & x_1^{k+1} & = \frac{1}{3}\left(2 - x_2^k - x_3^k\right) \\ i=2, & x_2^{k+1} & = \frac{1}{5}\left(17 - x_1^k - 2x_3^k\right) \\ i=3, & x_3^{k+1} & = -\frac{1}{6}\left(-18 - 2x_1^k + x_2^k\right) \end{cases}$$

Starting from $X^0 = (0,0,0)^t$, we obtain

$$\begin{aligned} X_1 &= (1.5000, 0.8571, 4.2500)^t \\ X_2 &= (-0.1964, 2.4643, 3.9821)^t \\ X_3 &= (0.7411, 2.5918, 2.3527)^t \\ X_4 &= (1.6196, 1.7596, 2.4914)^t \\ X_5 &= (1.1341, 1.6935, 3.3352)^t \\ X_6 &= (0.6791, 2.1245, 3.2634)^t \end{aligned}$$

with $\varepsilon = \|X_6 - X^*\| = 0.4334$.
- For each iteration k, the iterative scheme of the Gauss-Seidel method is written in this case as follows :

$$\begin{cases} i = 1, \quad x_1^{k+1} &= \frac{1}{3}\left(2 - x_2^k - x_3^k\right) \\ i = 2, \quad x_2^{k+1} &= \frac{1}{5}\left(17 - x_1^{k+1} - 2x_3^k\right) \\ i = 3, \quad x_3^{k+1} &= -\frac{1}{6}\left(-18 - 2x_1^{k+1} + x_2^{k+1}\right) \end{cases}$$

Starting from $X^0 = (0,0,0)^t$, we obtain

$$\begin{aligned} X_1 &= (1.5, 0.6429, 4.1429)^t \\ X_2 &= (-0.25, 2.6684, 2.1862)^t \\ X_3 &= (1.7411, 1.5454, 3.5262)^t \\ X_4 &= (0.5096, 2.2956, 2.6557)^t \\ X_5 &= (1.3199, 1.8067, 3.2249)^t \\ X_6 &= (0.7909, 2.1263, 2.8530)^t \end{aligned}$$

avec $\varepsilon = \|X_6 - x^*\| = 0.2851$.
b- We note that, for the same number of iterations, the approximate solution obtained by the Gauss-Seidel method is more precise.

Exercice 3. Using the Gauss-Seidel method, approximate the solution of the following system of linear equation within a precision of 10^{-3}

$$\begin{cases} 8x_1 + x_2 + x_3 &= 26 \\ x_1 + 5x_2 - x_3 &= 7 \\ x_1 - x_2 + 5x_3 &= 7 \end{cases}$$

Solution .
For each iteration k, the Gauss-Seidel method is written in this case as follows :

$$\begin{cases} i = 1, \quad x_1^{k+1} &= \frac{1}{8}\left(26 - x_2^k - x_3^k\right) \\ i = 2, \quad x_2^{k+1} &= \frac{1}{5}\left(7 - x_1^{k+1} + x_3^k\right) \\ i = 3, \quad x_3^{k+1} &= \frac{1}{5}\left(7 - x_1^{k+1} + x_2^{k+1}\right) \end{cases}$$

Starting from $X^0 = (0,0,0)^t$, it results that

$$\begin{aligned} X_1 &= (3.25, 0.75, 0.9000)^t \\ X_2 &= (3.0438, 0.9712, 0.9855)^t \\ X_3 &= (3.0054, 0.996, 0.9981)^t \\ X_4 &= (3.0007, 0.9995, 0.9997)^t \\ X_5 &= (3.0001, 0.9999, 1)^t . \end{aligned}$$

The generated vectors of this system converge to $X^* = (3, 1, 1)^t$.