Chapter 2

Structure of real numbers field $\mathbb{R}$ Dr L.Derbal

The aim of this chapter is to introduce axiomatically the set of Real numbers

2.1 Set of rational numbers $\mathbb{Q}$.

2.1.1 Integers numbers

We take for granted the system $\mathbb{N}$ of natural numbers $\mathbb{N} = \{1, 2, 3, 4...\}$. In general the equation $x + a = 0$ is not solvable in $\mathbb{N}$ whose case or a is positive. In order to make this equation solvable, we must enlarge the set $\mathbb{N} = \mathbb{Z}_+$ by introducing negative integers as unique solutions of the equations $a + x = 0$ (existence of the additive inverse) for each $a \in \mathbb{N}$. Our extended system, which is denoted by $\mathbb{Z}$, now contains all integers and can be arranged in order

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\} = \mathbb{N} \cup \{0\} \cup \{-a \ : a \in \mathbb{N}\}.$$

theorem 2.1.1 (Fundamental theorem of arithmetic) *Every positive integer except* 1 *can be expressed uniquely as a product of primes.*

2.1.2 Rational Numbers

Let $a \in \mathbb{Z}$, $b \in \mathbb{Z}$. The equation

$$ax = b \tag{1}$$

need not have a solution $x \in \mathbb{Z}$. In order to solve (1) (for $a \neq 0$) we have to enlarge our system of numbers again so that it includes fractions $\frac{b}{a}$ (existence of multiplicative inverse in $\mathbb{Z}-\{0\}$.This motivates the following definition.

Definition 2.1.2 *The set of rational numbers (or rationals) $\mathbb{Q}$ is the set*

$$\mathbb{Q}=\{r=\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, hcf(p,q)=1\}.$$

Here $hcf(p,q)$ stands for the highest common factor of p and q, so when writing $\frac{p}{q}$ for a rational we often assume that the numbers p and q have no common factor greater than 1.

Definition 2.1.3 *Let $b \in \mathbb{N}$, $d \in \mathbb{N}$. Then*

$$\left(\frac{a}{b} > \frac{c}{d}\right) \Leftrightarrow (ad > bc)$$

The following theorem provides a very important property of rationals.

theorem 2.1.4 *Between any two rational numbers there is another (and, hence, infinitely many others).*

proof. Let $b \in \mathbb{N}$, $d \in \mathbb{N}$, and $\frac{a}{b} > \frac{c}{d}$.
Notice that

$$(\forall m \in \mathbb{N}) \left[\frac{a}{b} > \frac{a+mc}{b+md} > \frac{c}{d}\right].$$

Indeed, since b, d and m are positive we have

$$[a(b+md) > b(a+mc)] \Leftrightarrow [mad > mbc] \Leftrightarrow (ad > bc),$$

and

$$[d(a+mc) > c(b+md)] \Leftrightarrow (ad > bc).$$

■

2.2 Irrational Numbers

Suppose that $a \in Q^{+}$ and consider the equation

$$x^2 = a. \tag{2}$$

In general (2) does not have rational solutions. For example, the following theorem holds.

theorem 2.2.1 *No rational number has square* 2.

proof. Suppose for a contradiction that the rational number $\frac{p}{q}$, $(p \in \mathbb{Z}, q \in \mathbb{N}$, in lowest terms) is such that $(\frac{p}{q})^2 = 2$. Then $p^2 = 2q^2$.

Hence, appealing to the Fundamental Theorem of Arithmetic, p^2 is even, and hence p is even. Thus $(\exists k \in \mathbb{Z})$ $[p = 2k]$. This implies that

$$2k^2 = q^2,$$

and therefore q is also even. The last statement contradicts our assumption that p and q have no common factor. ■

The last theorem provides an example of a number which is not rational. We call such numbers irrational.

We leave the following as an exercise.

Exercise 2.2.2 *No rational x satisfies the equation*

- $x^3 = x + 7$.
- $x^5 = x + 4$.

2.3 Real numbers

Real numbers can be defined as the union of both rational and irrational numbers. They can be both positive or negative and are denoted by the symbol "$\mathbb{R}$". All the natural numbers, decimals and fractions come under this category.In this course we postulate the existence of the set of real numbers $\mathbb{R}$ as well as basic properties summarized in a collection of axioms.Will find that axioms $A.1 - A.11$ characterize $\mathbb{R}$ as an algebraic field.

2.3.1 Axiomatic definition

$A.1$ $\forall a, b \in \mathbb{R} : (a + b) \in \mathbb{R}$ (closed under addition).

$A.2$ $\forall a, b \in \mathbb{R} : [a + b = b + a]$ (commutativity of addition).

$A.3$ $\forall a, b, c \in R : [(a + b) + c = a + (b + c)]$ (associativity of addition).

$A.4$ $\exists 0 \in \mathbb{R}, \forall a \in \mathbb{R} : [0 + a = a]$ (existence of additive identitie).

$A.5$ $\forall a \in \mathbb{R}, \exists! x \in \mathbb{R} : [a + x = 0]$ (existence of additive inverse). We write $x = -a$.

Axioms $A.6 - A.10$ are analogues of $A.1 - A.5$ for the operation of multiplication.

$A.6$ $\forall a, b \in \mathbb{R} : [ab \in \mathbb{R}]$ (closed under multiplication).

$A.7$ $\forall a, b \in \mathbb{R} : [ab = ba]$ (commutativity of multiplication).

$A.8$ $\forall a, b, c \in \mathbb{R} : [(ab)c = a(bc)]$ (associativity of multiplication).

$A.9$ $\exists 1 \in \mathbb{R}, \forall a \in \mathbb{R} : [1 \cdot a = a]$ (existence of multiplicative identitie).

$A.10$ $\forall a \in \mathbb{R} - \{0\}, \exists! y \in \mathbb{R} : [ay = 1]$ (existence of multiplicative inverse). We write $y = \frac{1}{a}$.

The last axiom links the operations of summation and multiplication.

A.11 $\forall a, b, c \in \mathbb{R} : [(a + b)c = ac + bc]$ (distributive law).

Example 2.3.1 $\forall a \in \mathbb{R} : 0a = 0]$.

Indeed, we have

$$\begin{aligned} a + 0a &= 1a + 0a \text{ (by A.9)} \\ &= (1 + 0)a \text{(by A.11)} \\ &= 1a \text{ (by A.2 and A.4)} \\ &= a \text{ (by A.9)} \end{aligned}$$

Now add $-a$ to both sides.

$$\begin{aligned} -a + (a + 0a) &= -a + a \\ &\Rightarrow (-a + a) + 0a = 0 \text{ (by A.3 and A.5)} \\ &\Rightarrow 0 + 0a = 0 \text{ (by A.5)} \\ &\Rightarrow 0a = 0 \text{ (by A.4)}. \end{aligned}$$

Remark 2.3.2 *The set of rationals $\mathbb{Q}$ also forms an algebraic field (that is, the rational numbers satisfy axioms **A.1 - A.11**).*

Now we add axioms of order.

O.1 $\forall a, b \in \mathbb{R} : [(a = b) \vee (a < b) \vee (a > b)]$

$$\equiv \forall a, b \in \mathbb{R} : [(a \geq b) \wedge (b \geq a) \Rightarrow (a = b)] \text{ (trichotomy law)}.$$

O.2 $\forall a, b, c \in \mathbb{R} : [(a > b) \wedge (b > c) \Rightarrow (a > c)]$ (transitive law).

O.3 $\forall a, b, c \in \mathbb{R} : [(a > b) \Rightarrow (a + c > b + c)]$ (compatibility with addition).

O.4 $\forall a, b, c \in \mathbb{R} : [(a > b) \wedge (c > 0) \Rightarrow (ac > bc)]$ (compatibility with multiplication).

Remark 2.3.3 *Note that*

$$\forall a, b \in \mathbb{R} : \{(a > b) \Leftrightarrow (a - b > 0)\}.$$

*This follows from (**O.3**) by adding $-b$.*

Axioms **A.1-A.11** and **O.1 - O.4** define $\mathbb{R}$ to be an ordered field. Observe that the rational numbers also satisfy axioms **A.1 - A.11** and **O.1 - O.4**, so $\mathbb{Q}$ is also an ordered field.

2.3.2 Absolute value

Definition 2.3.4 *We define the maximum and the minimum of two real a and b by:*

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}, \quad \min(a, b) \begin{cases} a & \text{if } a \leq b \\ b & \text{if } b < a \end{cases}$$

Definition 2.3.5 *The absolute value $|x|$ of x is defined by*

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0, \\ -x & \text{if} \quad x < 0. \end{cases}$$

theorem 2.3.6 *We can prove a bunch of theorems about the absolute value function that we usually take for granted:*

1) $|x| \geq 0$ *and* $(|x| = 0 \Leftrightarrow x = 0)$.
2) $\forall x \in \mathbb{R}$, $|-x| = |x|$.
3) $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$.
4) $|x^2| = x^2 = |x|^2$.
5) If $x, y \in \mathbb{R}$, *then* $|x| \leq y \Leftrightarrow -y \leq x \leq y$.
6) $\forall x \in \mathbb{R}$, $x \leq |x|$.

proof. :
1) If $x \geq 0$ then $|x| = x \geq 0$. If $x \leq 0$, then $-x \geq 0 \Rightarrow |x| = -x \geq 0$. Thus, $|x| \geq 0$. Now suppose $x = 0$.Then, $|x| = x = 0$. For the other direction, suppose $|x| = 0$. Then, if $x \geq 0 \Rightarrow x = |x| = 0$. If $x \leq 0$, then $-x = |x| = 0$. Therefore, $x = 0 \Leftrightarrow |x| = 0$.
2) If $x \geq 0$ then $-x \leq 0$. Thus, $|x| = x = -(-x) = |-x|$. If $x \leq 0$ then $-x \geq 0$ and thus $|-x| = |-(-x)| = |x|$.

3) a) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.
b) If $x \leq 0$ and $y \leq 0$, then $xy \geq 0 \Rightarrow |xy| = xy = (-x)(-y) = |x||y|$.
c) If $x \leq 0$ and $y \geq 0$, then $xy \leq 0 \Rightarrow |xy| = -xy = (-x)(y) = |x||y|$.
d) If $x \geq 0$ and $y \leq 0$, then $xy \leq 0 \Rightarrow |xy| = -xy = (x)(-y) = |x||y|$.
4) Take $x = y$ in 3). Then, $|x^2| = |x|^2$. Since $x^2 \geq 0$, it follows that $|x^2| = x^2$.
5) Suppose $|x| \leq y$. If $x \geq 0$, then $-y \leq 0 \leq x = |x| \leq y$. Therefore, $-y \leq x \leq y$.
If $x \leq 0$, then $-x \geq 0$ and $|x| = -x \leq y$. Hence, $-y \leq -x \leq y \Rightarrow -y \leq x \leq y$.
6) If $x \geq 0$ then $x = |x|$. If $x \leq 0$ then $x \leq |x|$ and thus $x \leq |x|$. ■

theorem 2.3.7 (triangle inequality)

$$\forall a, b \in \mathbb{R} : |a + b| \leq |a| + |b|.$$

proof. We split the proof into two cases. We use the fact that $a \leq |a|$ for all $a \in \mathbb{R}$.
Case $a + b \geq 0$. Then

$$|a + b| = a + b \leq |a| + |b|.$$

Case $a + b < 0$. Then

$$|a + b| = -(a + b) = (-a) + (-b) \leq |a| + |b|.$$

■

Example 2.3.8 *Prove that*

1) $(\forall a \in \mathbb{R})(\forall b \in \mathbb{R})[a^2 + b^2 \geq 2ab]$.
2) $(\forall a \in \mathbb{R}^+)(\forall b \in \mathbb{R}^+)[\frac{a+b}{2} \geq \sqrt{ab}]$.
3) $(\forall a \in \mathbb{R}^+)(\forall b \in \mathbb{R}^+)(\forall c \in \mathbb{R}^+)(\forall d \in \mathbb{R}^+)\left[\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}\right]$.
4) Let $n \geq 2$ be a natural number. Prove that

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{1}{2}.$$

Recall that $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

proof. 1) The result is equivalent to $a^2 + b^2 - 2ab \geq 0$. But,

$$a^2 + b^2 - 2ab = (a - b)^2 \geq 0.$$

Note that the equality holds if and only if $a = b$.
2) As above, let us prove that the difference between the left-hand side (LHS)

and the right-hand side (RHS) is non-negative:

$$\frac{a+b}{2}-\sqrt{ab}=\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}\geq 0.$$

The equality holds if and only if $a=b$.

3)By (**2**) and by (**O.2**) we have

$$\frac{a+b+c+d}{4}\geq\frac{\sqrt{ab}+\sqrt{cd}}{2}\geq\sqrt{\sqrt{ab}\sqrt{cd}}=\sqrt[4]{abcd}.$$

The equality holds if and only if $a=b=c=d$.

4) $\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}>\underbrace{\frac{1}{2n}+\frac{1}{2n}+\cdots+\frac{1}{2n}}_{n}=\frac{n}{2}=\frac{1}{2}.$ ■

theorem 2.3.9 (Bernoulli's inequality) $\forall n\in\mathbb{N},\forall x>-1:[(1+x)^n\geq 1+xn]$.

proof. Base case. The inequality holds for $n=0,1$.

Induction step. Suppose that the inequality is true for $n=k$ with $k\geq 1$; that is,

$$(1+x)^k\geq 1+kx.$$

We have to prove that it is true for $n=k+1$; in other words,

$$(1+x)^{k+1}\geq 1+(k+1)x.$$

Now,

$$\begin{aligned}(1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x)\\ &= 1+(k+1)x+kx^2\\ &\geq 1+(k+1)x.\end{aligned}$$

This concludes the induction step. By the principle of mathematical induction, the result is true for all $n\in\mathbb{N}$. ■

2.3.3 Bounded Sets of $\mathbb{R}$

Definition 2.3.10 *Let A be a subset of $\mathbb{R}$ and non-empty .*

We say that A is bounded from above if and only if :

$$\exists M\in\mathbb{R};\forall x\in A: x\leq M$$

We say that A is bounded from below if and only if

$$\exists m \in \mathbb{R}; \forall x \in A : x \geq m$$

A is bounded if and only if it is bounded from above and below.

Proposition 2.3.11 *The three following conditions are equivalent*

1) A is bounded set,

2) $\exists m \in \mathbb{R}, \exists M \in \mathbb{R}; \forall x \in A : m \leq x \leq M$.

3) $\exists M \in \mathbb{R}_+^*; \forall x \in A : |x| \leq M$.

Definition 2.3.12 *Let $A \subseteq \mathbb{R}$. We say that $M \in \mathbb{R}$ is the supremum of A, written* $\sup A$*, if*

(i) $\forall x \in A : x \leq M$ for all $x \in A$; (M is an upper bound of A)

(ii) if $x \leq \grave{M}$ for all $x \in A$ then $M \leq \grave{M}$ (M is the least upper bound of A).

Definition 2.3.13 *Let $A \subseteq \mathbb{R}$. We say that $m \in \mathbb{R}$ is the infimum of A, written* $\inf A$*, if*

(i) $\forall x \in A : x \geq m$ for all $x \in A$; (m is a lower bound of A)

(ii) if $x \geq \grave{m}$ for all $x \in A$ then $m \geq \grave{m}$ (m is the greatest lower bound of A).

Definition 2.3.14 *If* $\sup A \in A$*, it is called* $\max A$.

If $\inf A \in A$*, it is called* $\min A$.

Notation 3 *If A is infinite from above (from below, respectively) in $\mathbb{R}$ we write* $\sup A = +\infty$ *(*$\inf A = -\infty$*,respectively).*

Remark 2.3.15 *If A has a supremum (an infimum, respectively), then* $\sup A$ *(*$\inf A$*) is unique.*

Example 2.3.16 • *Let $A = [1, 2)$. Then 2 is an upper bound, and is the least upper bound: if $\grave{M} < 2$ then $\grave{M}$ is not an upper bound because* $\max(1, 1 + \frac{\grave{M}}{2}) \in A$ *and* $\max(1, 1 + \frac{\grave{M}}{2}) > \grave{M}$*. Note that in this case* $\sup A \notin A$*, so* $\nexists \max A$*.*

• *Let $A = (1, 2]$. Then we again have* $\sup A = 2$*, and this time* $\sup A \in A$*. The supremum is the least upper bound of a set. There's an analogous definition for lower bounds.*

Axiom 2.3.17 (supermum and infimum) *Let A be a non-empty subset of $\mathbb{R}$ that is bounded above (below, respectively). Then A has a supremum (an infimum, respectively).*

Let's explore some useful properties of sup and inf.

Proposition 2.3.18 *(i) Let A, B be non-empty subsets of $\mathbb{R}$, with $A \subseteq B$ and with B bounded above. Then A is bounded above, and* $\sup A \leq \sup B$.

(ii) Let $B \subseteq \mathbb{R}$ be non-empty and bounded below. Let $A = \{-x : x \in B\}$. Then A is non-empty and bounded above. Furthermore, $\inf B$ *exists, and* $\inf B = -\sup A$.

proof. (i) Since B is bounded above, it has an upper bound, say M. Then $x \leq M$ for all $x \in B$, so certainly $x \leq M$ for all $x \in A$, so M is an upper bound for A. Now A, B are non-empty and bounded above, so by Axiom of supermum .

Note that $\sup B$ is an upper bound for B and hence also for A, so $\sup B \geq \sup A$ (since $\sup A$ is the least upper bound for A).

(ii) Since B is non-empty, so is A.

Let m be a lower bound for B, so $x \geq m$ for all $x \in B$. Then $-x \leq -m$ for all $x \in B$, so $y \leq -m$ for all $y \in A$, so $-m$ is an upper bound for A.

Now A is non-empty and bounded above, so by Axiom of supremum. Then $y \leq \sup A$ for all $y \in A$, so $x \geq -\sup A$ for all $x \in B$, so $-\sup A$ is a lower bound for B. Also, we saw before that if m is a lower bound for B then $-m$ is an upper bound for A. Then $-m \geq \sup A$ (since $\sup A$ is the least upper bound), so $m \leq -\sup A$.

So $-\sup A$ is the greatest lower bound.

So $\inf B$ exists and $\inf B = -\sup A$. ■

Proposition 2.3.19 (Approximation property) *1) Let $A \subseteq \mathbb{R}$ be non-empty and bounded above, then*

$$M = \sup A \Leftrightarrow \begin{cases} \forall x \in A : x \leq M \\ and \\ \forall \epsilon ; \exists a_\epsilon \in A : M - \epsilon < a_\epsilon \end{cases}$$

2)Let A be bounded from below, then

$$m = \inf A \Leftrightarrow \begin{cases} \forall x \in A : x \geq m \\ and \\ \forall \epsilon ; \exists b_\epsilon \in A : b_\epsilon < m + \epsilon \end{cases}$$

proof. 1) Take $\epsilon > 0$. Note that by definition of the supremum we have $x \leq \sup A$ for all $x \in A$. Suppose, for a contradiction, that $\sup A - \epsilon \geq x$ for all $x \in A$. Then $\sup A - \epsilon$ is an upper bound for A, but $\sup A - \epsilon < \sup A$. Contradiction.

So there is $a_\epsilon \in A$ with $\sup A - \epsilon < a_\epsilon$.

2) In the same way we prove the second case. ■

Axiom 2.3.20 (of Archimedes) $\forall x > 0$; $\forall y \in \mathbb{R}$; $\exists n \in \mathbb{N}^* : y < nx$.

proof. We suppose that: $\exists x > 0$; $\exists y \in \mathbb{R}$; $\forall n \in \mathbb{N}^* : y \geq nx$ or $\exists x > 0$; $\exists y \in \mathbb{R}$; $\forall n \in \mathbb{N}^* : n \leq \frac{y}{x}$, that's mean the set $\mathbb{N}^*$ is limited from above it accepts an upper limit in $\mathbb{R}$ called M. so

$$\forall \epsilon; \exists n_\epsilon \in \mathbb{N}^* : M - \epsilon < n_\epsilon.$$

Putting $\epsilon = 1$, we get :

$$\exists n_\epsilon \in \mathbb{N}^* : M - 1 < n_\epsilon \text{ or } \exists n_\epsilon \in \mathbb{N}^* : M < n_\epsilon + 1$$

but $n_\epsilon + 1 \in \mathbb{N}^*$, this is a contradiction with $\sup \mathbb{N}^* = M$. ■

Example 2.3.21 $A = [1, 2[$; $\sup A = 2 \notin A$, *then* $\nexists \max A$; $\inf A = 1 = \min A$

$A = \left\{ \frac{1}{n}; n \in \mathbb{N}^* \right\}$; $\forall n \in \mathbb{N}^* : n \geq 1 \Rightarrow 0 < \frac{1}{n} \leq 1$, *then* $\sup A = \max A = 1 \in A$.

Let we proof that $\inf A = 0$ *i.e.*

$$0 = \inf A \Leftrightarrow \begin{cases} \forall x \in A : x \geq 0 \\ \text{and} \\ \forall \epsilon, \exists a_\epsilon \in A : a_\epsilon < 0 + \epsilon \end{cases}$$

On the other side we have

$$\forall \epsilon, \exists a_\epsilon \in A : a_\epsilon < 0 + \epsilon \Leftrightarrow \forall \epsilon, \exists n \in \mathbb{N}^* : \frac{1}{n} < \epsilon.$$

and this proposition is true and its according to Archimedes' Axiom

$$\forall \epsilon, \exists n \in \mathbb{N}^* : n\epsilon > 1$$

$\min A =$ *unavailable, because* $0 \notin A$.

Definition 2.3.22 *Let* $x \in \mathbb{R}$, *there exists a unique relative integer, the integer part denoted* $E(x)$, *such that* $E(x) \leq x < E(x) + 1$. *We also note* $E(x) = [x]$.

Example 2.3.23 *1)* $E(3,5) = 3$ *since* $3 \leq 3,5 < 3 + 1$.

2) $E(-3,5) = -4$ *since* $-4 \leq -3,5 < -4 + 1$.

3) $\forall n \in \mathbb{N}^* : E\left(\frac{1}{n+1}\right) = 0$ *since* $\forall n \in \mathbb{N}^* : 0 \leq \frac{1}{n+1} < 0 + 1$.

2.3.4 Dense groups in $\mathbb{R}$

theorem 2.3.24 *Between every two different real numbers there is at least one rational number.*

proof. Let y and x be two real numbers where $x < y$. According to Archimedean axiom

$$\exists n \in \mathbb{N}^* : 1 < n(y-x) \text{ or } nx + 1 < ny.$$

On the other hand we have

$$\begin{aligned} E(nx) &\leq nx < E(nx) + 1 \\ \text{or } nx &< E(nx) + 1 \leq nx + 1 < ny. \end{aligned}$$

So

$$x < \frac{E(nx)+1}{n} < y$$

Well the rational number $\frac{E(nx)+1}{n}$ is bounded between the two real numbers x, y. ■

theorem 2.3.25 *between every two different real numbers there is at least one irrational number.*

To prove this theory we need the following proposition.

Proposition 2.3.26 *if $x \in I$ (irrational number)and $r \in \mathbb{Q}^*$ then $rx \in I$.*

proof. We assume $x \in I$ and $r \in \mathbb{Q}^*$and that $rx \in \mathbb{Q}$, then

$$\begin{aligned} \left(\frac{1}{r} \in \mathbb{Q}^* \text{or } rx \in \mathbb{Q}\right) &\Rightarrow \frac{1}{r}.rx \in \mathbb{Q} \\ &\Rightarrow x \in \mathbb{Q}. \end{aligned}$$

This is a contradiction because $x \in I$. ■

theorem. Let x, y be two real numbers, where $x < y$, according to the theorem , there sexist a rational number r $(r \neq 0)$ such that:

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}} \text{ or } x < r\sqrt{2} < y$$

and according to proposition we conclude that $r\sqrt{2}$ is a irrational number. ■

Corollary 2.3.27 *The two sets $\mathbb{Q}$ and I is dense in $\mathbb{R}$.*

2.3.5 Intervals in $\mathbb{R}$

Definition 2.3.28 *An interval is a subset of the real numbers that contains all real numbers lying between any two numbers of the subset.*

Let a, b be two real numbers, where $a < b$, we define

- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is called closed interval.
- $]a, b[= \{x \in \mathbb{R} : a < x < b\}$ is called open interval.
- $[a, b[= \{x \in \mathbb{R} : a \leq x < b\}$ is called half open interval.
- $]a, b] = \{x \in \mathbb{R} : a < x \leq b\}$" " " " " " " " " " " " " " " " ".
- $[a, +\infty[= \{x \in \mathbb{R} : x \geq a\}$ is unbounded closed interval.
- $]-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$" " " " " " " " " " " " " " " " ".
- $]a, +\infty[= \{x \in \mathbb{R} : x > a\}$ is unbounded open interval.
- $]-\infty, b[= \{x \in \mathbb{R} : x < b\}$" " " " " " " " " " " " " " " " ".
- $]-\infty, +\infty[$" " " " " " " " " " " " " " ".

Exercise 2.3.29 *Let*

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}^*\right\}.$$

Find $\inf S$ *and* $\sup S$ *and prove your answers.*

Solution 2.3.30 *We claim that* $\inf S = \frac{1}{2}$ *and* $\sup S = 2$. *Note that, if n is odd,* $1 - \frac{(-1)^n}{n} = 1 + \frac{1}{n}$,*while if n is even,* $1 - \frac{(-1)^n}{n} = 1 - \frac{1}{n}$.

It follows, if n is odd, that

$$1 - \frac{(-1)^n}{n} > 1 > \frac{1}{2}.$$

If $n \geq 2$ is even,

$$1 - \frac{(-1)^n}{n} = 1 - \frac{1}{n} \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Arguing similarly, $1 - \frac{(-1)^n}{n} \leq 2$ *and so* $\frac{1}{2}$ *and* 2 *are, respectively, lower and upper bounds for S. Since* $\frac{1}{2} \in S$, *there cannot be a lower bound* $m > \frac{1}{2}$ *and so* $\frac{1}{2}$*is the greatest lower bound for S, i.e.* $\inf S = \frac{1}{2}$. *Since* $2 \in S$, *there cannot be a upper bound* $M < 2$ *and so* 2*is the least upper bound for S, i.e.* $\sup S = 2$.

Exercise 2.3.31 *Let* $A = \left\{ \frac{1}{n} : \quad n = 1, 2, 3, ... \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$

Example 2.3.32 *1) Show that A is a non-empty set, both bounded above and below.*

2) Show that $\sup(A) \ = \max(A) = 1$.

3) Show that $\inf(A) \ = 0$.

4) Show that $\min(A)$ *does not exist..*

Solution 2.3.33 *Let* $A = \left\{ \frac{1}{n} : \quad n = 1, 2, 3, ... \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ... \right\}$

1) $1 \in A \Rightarrow A \neq \emptyset$, $\forall n : n \geq 1$ *we have* $0 < \frac{1}{n} \leq 1 \Rightarrow 1$ *is an upper bound of A and* 0 *is a lower bound of A.*

2) $\sup A$ *and* $\inf A$ *exist, according to the axiom of the upper bound : Let's show that* $\sup A = 1$. *Let* $\varepsilon > 0$, *we show that* $\exists x_0 \in A / \ x_0 > 1 - \varepsilon$. *In fact, let's take* $x_0 = 1$. *First of all* $x_0 = 1$ *verifies the precedent relation, since :* $\forall \varepsilon > 0$, $1 > 1 - \varepsilon$, *morever* $1 \in A$ *then :* $\sup A = \max A = 1$.

3) $\inf A = 0$? *Let* $\varepsilon > 0$, *we show that* $\exists x_0 \in A / \quad 0 + \varepsilon > x_0$, *the elements of A are of the form* $\frac{1}{n}$ *we must find* $n \in \mathbb{N}^* / \ \frac{1}{n} < \varepsilon$ *or* $n > \frac{1}{\varepsilon}$. *For* $\varepsilon > 0$ *if we take* $x_0 = \frac{1}{n}$ *with* $n > \frac{1}{\varepsilon}$ *we obtain* $x_0 \in A$ *and* $0 + \varepsilon > x_0$ *then* $\inf A = 0$.

4) We have $\forall n \geq 1, \frac{1}{n} > 0 \Rightarrow 0 \notin A \Rightarrow \nexists \min A$.