Setif1 University
Faculty of sciences.

TC-SM

Maths1
1^{st} guided works

Instruction. The questions mentioned by (*) are left to the students.
Exercise 1.

1 Consider the following sets

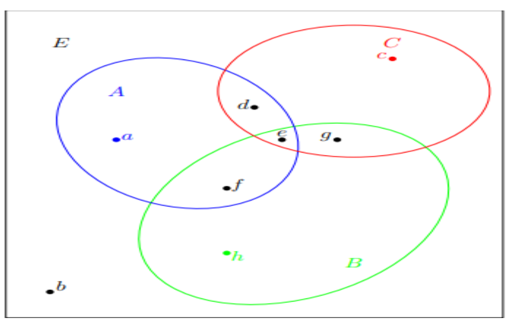
$$A = \{1, 3, 7, 9, 12\}, \;\; B = \{1, 3, 2\}, \;\; C = \{3, 4, 7, 9\}, \;\; D = \{3, 1\}.$$

Describe the following sets and their cardinals: $A \cap B$, $A \setminus B$, $A \Delta B$, $D \times C$, $*B \cap C$, $\complement_A(D)$, $*D \cup A$, $\mathcal{P}(C)$.

2 Describe the following sets:

$$F = [-2, 1[\cap] - \infty, 0], *E = [-2, 1[\cup] - \infty, 0], *G = [-2, 1[\Delta] - \infty, 0], *H = \complement_{\mathbb{R}}(F).$$

Exercise 2: Consider the following diagram, witch contains three subsets A, B, C of a set E and the elements a, b, c, d, e, f, g, h of E



Determine whether the following statements are true or false
1) $g \in A \cap \bar{B}$ 2) $g \in \bar{A} \cap \bar{B}$ 3) $g \in \bar{A} \cup \bar{B}$
4) $f \in \bar{A}$ 5) $e \in \bar{A} \cap \bar{B} \cap \bar{C}$.   6) $\{h, b\} \subset \bar{A} \cap \bar{B}$
7) $\{a, f\} \subset A \cup C$.

Exercise 3: If we have $C \subset A \cup B$, is it because $C \subset A$ or $C \subset B$?

Exercise 4: Let A, B, C be three subsets of the set E, for $X \subset E$, denoted by X^c the complement of X in E.
Prove the following Morgan's laws :
1. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ *2. $(A^c)^c = A$.

Exercise 5 : *Find the set of parts of the set $E = \{a, b, c, d\}$.

Exercise 6 : Let E and F be two sets, and let A and C be subsets of E and D, B be two subsets of F.Prove it

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

Exercise 7: Determine if the relations are reflexive, symmetric, transitive, or antisymmetric.
(1) $E = \mathbb{Z}$ and $x\mathcal{R}y \Leftrightarrow x = -y$
*(2) $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow \cos^2 x + \sin^2 y = 1$
*(3) $E = \mathbb{N}$ and $x\mathcal{R}y \Leftrightarrow \exists p, q \geq 1,\ y = px^q$. Where p and q are natural numbers.

Exercise 8: Let $\mathcal{R}$ be the relation defined on $\mathbb{R}^2$ by: $(x_1, y_1)\mathcal{R}(x_2, y_2) \Leftrightarrow y_1 = y_2$.
(1) Show that $\mathcal{R}$ is an equivalence relation.
(2) Determine the equivalence class of $(1, 0)$.
*(3) Same questions for the relation $\mathcal{R}$ defined on $\mathbb{R}^2$ by: $(x_1, y_1)\mathcal{R}(x_2, y_2) \Leftrightarrow x_1^2+y_1^2 = x_2^2+y_2^2$.

Exercise 9: Let $\mathcal{R}$ be the relation defined on $\mathbb{N}^*$ by : $n\mathcal{R}m \Leftrightarrow \exists k \in \mathbb{N}^* : n = km.$.
(1) Show that $\mathcal{R}$ is a relation of order
(2) Is the order total?

Exercise 10:

1. Let the function be $f : \mathbb{R} \to \mathbb{R}$, where $x \mapsto x^2$and let be.$A = [-1, 4]$ find :
   - The direct image of the set A by application f .
   - The reciprocal (inverse) image of the set A by application f.
2. *Let the function be $\sin : \mathbb{R} \to \mathbb{R}$.
   - What is the direct image of the set $\mathbb{R}$, the set $[0, 2\pi]$ and the set $[0, \pi/2]$?
   - What is the inverse image of set $[0, 1]$, the set $[3, 4]$ and the set $[1, 2]$?

Exercise 11: Show that the following functions are maps and then check whether they are injective, surjective or bijective
$f_1 : \mathbb{Z} \to \mathbb{Z},\ n \mapsto 2n,$ $\qquad f_2 : \mathbb{Z} \to \mathbb{N},\ n \mapsto 4n^2 + 5$
$f_3 : \mathbb{R} \to \mathbb{R},\ x \mapsto x^2,$ $\qquad f_4 : \mathbb{R} \to \mathbb{R}+, x \mapsto x^2.$

Exercise 12: Let the functions f and g defined from $\mathbb{N}$ to $\mathbb{N}$ by the following:

$$f(x) = 2x \text{ and } g(x) = \begin{cases} \dfrac{x}{2}, & \text{If } x \text{ is even} \\ 0, & \text{If } x \text{ is odd} \end{cases}$$

Find $g \circ f$ and $f \circ g$. Are the functions f and g injective, surjective or bijective?

Exercise 13: Show by recurrence

1. For any $n \in \mathbb{N} : 5n^3 + n$ is divisible by 6. (Indication $n\,(n + 1) = 2p,\ p \in \mathbb{N}$).
2. *For any $n \in \mathbb{N} : 2^n > n$.
3. *If $x \neq 1,\ x \in \mathbb{R},\ n \geq 1 : 1 + x + x^2 + ... + x^{n-1} = \dfrac{1 - x^n}{1 - x}$.

Correction of exercises

Exercise 1.

1 We have
$$A=\{1,3,7,9,12\},\ \ B=\{1,3,2\},\ \ C=\{3,4,7,9\},\ \ D=\{3,1\}.$$

Then $A\cap B=\{1,3\}$, Card $(A\cap B)=2$, $A\backslash B=\{7,9,12\}$, Card$(A\backslash B)=3$, $A\Delta B=(A\cup B)\backslash(A\cap B)=\{1,3,2,7,9,12\}\backslash\{1,3\}=\{2,7,9,12\}$, Card$(A\Delta B)=4$, $D\times C=\{3,1\}\times\{3,4,7,9\}=\{(3,3),(3,4),(3,7),(3,9),(1,3),(1,4),(1,7),(1,9)\}$, Card$(D\times\ C)=8$, $B\cap C=\{3\}$, Card$(B\cap C)=1$, $\complement_D(A)=\emptyset$, Card $\left(\complement_D(A)\right)=0$, $\complement_A(D)=\{7,9,12\}$, Card $\left(\complement_A(D)\right)=3$, $D\cup A=A$, $Card(D\cup A)=5$,

$$\mathcal{P}(C)=\left\{\begin{array}{l}\emptyset,\{3\},\{4\},\{7\},\{9\},\{3,4\},\{3,7\},\{3,9\},\{4,7\},\{4,9\},\{7,9\},\\ \{3,4,7\},\{3,4,9\},\{3,7,9\},\{4,7,9\},\{3,4,7,9\}\end{array}\right\},$$

$Card(\mathcal{P}(C))=2^{Card(C)}=2^4=16.$

2

$$F=[-2,1[\cap]-\infty,0],\ *E=[-2,1[\cup]-\infty,0],\ *G=[-2,1[\Delta]-\infty,0],\ *H=\complement_{\mathbb{R}}(F).$$

$F=[-2,1[\cap]-\infty,0]=[-2,0]$, $E=[-2,1[\cup]-\infty,0]=]-\infty,1[$,

$G=[-2,1[\Delta]-\infty,0]=[-2,1[\backslash]-\infty,0]\cup]-\infty,0]\setminus[-2,1[=]0,1[\cup]{-\infty},-2[$

$H=\complement_{\mathbb{R}}(F)=]-\infty,-2[\cup]0,+\infty[.$

Exercise 2.
1) false, 2) false, 3) true, 4) false, 5) false, 6) false:, 7) true.
Exercise 3: If we have $C\subset A\cup B$, is it because $C\subset A$ or $C\subset B$?
It is not true to take for example : $A=\{1,2\}$, $B=\{3,4\}$, $C=\{2,3\}$.
Exercise 4: Every time we prove double containment.
1. $(A\cap B)\cup C=(A\cup C)\cap(B\cup C)$
Let $x\in(A\cap B)\cup C$ and from it $x\in A$ and $x\in B$ or $x\in C$ If ($x\in A$ and $x\in B$), then $x\in A\cup C$ and $x\in B\cup C$, the containment is proved. If $x\in C$ in this case, we also have .$x\in A\cup C$ and $x\in B\cup C$.
On the other hand if $x\in A\cup C$ and $x\in B\cup C$, we distinguish between two cases: if $x\in C$, then .$x\in(A\cap B)\cup C$. Otherwise, $x\notin C$, but since $x\in A\cup C$ we have $x\in A$, similarly since $x\in B\cup C$ then $x\in B$ this proves that $x\in A\cap B$, consequently $x\in(A\cap B)\cup C$
2. $(A^c)^c=A$
Let $x\in(A^c)^c$ and from it $x\notin A^c\Rightarrow x\in A$.On the other hand if $x\in A$, then $x\notin A^c$, consequently $x\in(A^c)^c$.
3. $(A\cap B)^c=A^c\cup B^c$
Let $x\in(A\cap B)^c$ then $x\notin(A\cap B)$, as we have $x\notin A$ or $x\notin B$, that is $x\in A^c$ or $x\in B^c$. We conclude that $x\in A^c\cup B^c$. On the other hand if $x\in A^c\cup B^c$ then $x\in A^c$ or $x\in B^c$, that is, $x\notin A$ or $x\notin B$, inparticular, $x\notin A\cap B$, consequently $x\in(A\cap B)^c$
4.We can also introduce the above logic in the equivalence model.

$$
\begin{aligned}
x \in (A \cup B)^c \ &\Leftrightarrow x \notin A \cup B \\
&\Leftrightarrow x \notin A \text{ and } x \notin B \\
&\Leftrightarrow x \in A^c \text{ and } x \in B^c \\
&x \in A^c \cap B^c
\end{aligned}
$$

Exercise 5: $E = \{a, b, c, d\}$

$$\mathcal{P}(E) = \left\{ \begin{array}{l} \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \\ \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\} \end{array} \right\}$$

Exercise 6: To prove the equality $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$., we need to show that each side is a subset of the other.
Step 1: Show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$.
Let $(x, y) \in (A \times B) \cap (C \times D)$.By definition of intersection, this means: $(x, y) \in A \times B$ and $(x, y) \in C \times D$
From $(x, y) \in A \times B$, we have $x \in A$ and $y \in B$ From $(x, y) \in C \times D$, we have $x \in C$ and $y \in D$
Since $x \in A$ and $x \in C$, it follows that $x \in A \cap C$. Similarly, since $y \in B$ and $y \in D$, it follows that $x \in B \cap D$.
Thus, we conclude that $(x, y) \in (A \cap C) \times (B \cap D)$. Therefore, we have shown:

$$(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$$

Step 2: Show that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$
Let $(x, y) \in (A \cap C) \times (B \cap D)$. By definition of the Cartesian product, this means: $x \in (A \cap C)$ and
$y \in (B \cap D)$. From $x \in A \cap C$, we have: $x \in A$ and $x \in C$. From $y \in (B \cap D)$, we have: $y \in B$ and $y \in D$.
Thus, we conclude that: $(x, y) \in A \times B$ and $(x, y) \in C \times D$.Therefore, we can conclude that $(x, y) \in (A \times B) \cap (C \times D)$. Thus, we have shown: $(A \cap C) \times (B \cap D \subseteq (A \times B) \cap (C \times D)$ .Since we have shown both inclusions:, we conclude that:
$(A \cap C) \times (B \cap D = (A \times B) \cap (C \times D)$. This completes the proof.

Exercise 7: (1) $E = \mathbb{Z}$ and $x\mathcal{R}y \Leftrightarrow x = -y$
The relation is not reflexive, because 1 is not in relation with itself. Indeed, $1 \neq -1$.The relationship is symmetrical, because $x = -y \Leftrightarrow y = -x$.It is not antisymmetric, because $1\mathcal{R}-1$ and $-1\mathcal{R}1$ while $-1\neq 1$.It is not transitive. Indeed, we have $1\mathcal{R}-1$ and $-1\mathcal{R}1$ and 1 and 1 are not related. This relationship is neither an equivalence relationship nor an order relationship.
(2) $E = \mathbb{R}$ and $x\mathcal{R}y \Leftrightarrow \cos^2 x + \sin^2 y = 1$.From the formula $\cos^2 x + \sin^2 x = 1$, we deduce that the relation is reflexive. It is also symmetrical. Indeed, if $x\mathcal{R}y$, i.e. $\cos^2 x + \sin^2 y = 1$, so we have $\cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y = \left(\cos^2 x + \sin^2 y\right) + \left(\cos^2 y + \sin^2 x\right) = 1 + \cos^2 y + \sin^2 x$ On the one hand, and $\cos^2 x + \sin^2 x + \cos^2 y + \sin^2 y = 2$,on the other hand, which leads well $\cos^2 y + \sin^2 x = 1$ and therefore the relation is symmetric. It is not antisymmetric, because $0\mathcal{R}2\pi$ and $2\pi\mathcal{R}0$,while $0 \neq 2\pi$. It is transitive. If $x\mathcal{R}y$ and $y\mathcal{R}z$, we have $\cos^2 x + \sin^2 y = 1$ and $\cos^2 y + \sin^2 z = 1$, either by adding $\cos^2 x + \left(\sin^2 y + \cos^2 y\right) + \sin^2 z = 2$, which implies $\cos^2 x + \sin^2 z = 1 \Leftrightarrow x\mathcal{R}z$. We are therefore dealing with an equivalence relationship.

(3) $E=\mathbb{N}$ and $x\mathcal{R}y \Leftrightarrow \exists p,q \geq 1,\ y=px^q$. Where p and q are natural numbers.
The relationship is reflexive (taking $p=q=1$), it is not symmetrical because if $x\mathcal{R}y$, we necessarily have $x \leq y$ So, we have $2\mathcal{R}4$ (take $p=2, q=1$), while we don't have $4\mathcal{R}2$. The relation is antisymmetric: if $x\mathcal{R}y$ and $y\mathcal{R}z$, , so we have $x \leq y$ and $y \leq x$ and so $x=y$. Finally, the relation is transitive. If $x\mathcal{R}y$ and $y\mathcal{R}z$, then there exist integers $p,q,a,b \geq 1$ such as $y=px^q$and $z=ay^b$.We deduce from this

$$z = a\left(px^q\right)^b = \left(ap^b\right)x^{bq}$$

and so $x\mathcal{R}z$ The relationship is a relationship of order.

Exercise 8:

(1) Let $\mathcal{R}$ be the relation defined on $\mathbb{R}^2$ by: $(x_1,y_1)\mathcal{R}(x_2,y_2) \Leftrightarrow y_1=y_2$.

· Let $(x_1,y_1)\in\mathbb{R}^2$, then $y_1=y_1 \Leftrightarrow (x_1,y_1)\mathcal{R}\ (x_1,y_1)$, so $\mathcal{R}$ is a reflexive relation

· Let $(x_1,y_1),\ (x_2,y_2)\in\mathbb{R}^2$, we have $(x_1,y_1)\mathcal{R}(x_2,y_2) \Leftrightarrow y_1=y_2 \Leftrightarrow y_2=y_1 \Leftrightarrow (x_2,y_2)\mathcal{R}(x_1,y_1)$

then $\mathcal{R}$ is a symmetric relationship.

· Let $(x_1,y_1),\ (x_2,y_2),(x_3,y_3)\in\mathbb{R}^3$, we have

$$\begin{aligned}[(x_1,y_1)\mathcal{R}(x_2,y_2) \text{ and } (x_2,y_2)\mathcal{R}(x_3,y_3)] &\Rightarrow [y_1=y_2 \text{ and } y_2=y_3]\\ &\Rightarrow [y_1=y_3]\\ &\Rightarrow (x_1,y_1)\mathcal{R}(x_3,y_3),\end{aligned}$$

therefore $\mathcal{R}$ is a transitive relation. We deduce that $\mathcal{R}$ is an equivalence relation.

(2) The equivalence class of $(1,0)$, we have

$$\begin{aligned}\mathcal{C}\left((1,0)\right) &= \left\{(x,y)\in\mathbb{R}^2 : (x,y)\,\mathcal{R}\,(1,0)\right\}\\ &= \left\{(x,y)\in\mathbb{R}^2 : y=0\right\} = \mathbb{R}\times\{0\}\end{aligned}$$

(3) Let the relation $\mathcal{R}$ be denied on $\mathbb{R}^2$ by :

$$(x_1,y_1)\mathcal{R}(x_2,y_2) \Leftrightarrow x_1^2+y_1^2 = x_2^2+y_2^2$$

· Let $(x_1,y_1)\in\mathbb{R}^2$, we have $x_1^2+y_1^2=x_1^2+y_1^2 \Leftrightarrow (x_1,y_1)\mathcal{R}\ (x_1,y_1)$, so $\mathcal{R}$ is a reflexive relation.

· Let $(x_1,y_1),\ (x_2,y_2)\in\mathbb{R}^2$, we have

$$\begin{aligned}(x_1,y_1)\mathcal{R}(x_2,y_2) &\Leftrightarrow \left(x_1^2+y_1^2=x_2^2+y_2^2\right)\\ &\Leftrightarrow \left(x_2^2+y_2^2=x_1^2+y_1^2\right)\\ &\quad (x_2,y_2)\mathcal{R}(x_1,y_1),\end{aligned}$$

then $\mathcal{R}$ is a symmetric relationship.

· Let $(x_1, y_1),\ (x_2, y_2), (x_3, y_3) \in \mathbb{R}^3$, we have

$$\begin{aligned} [(x_1,y_1)\mathcal{R}(x_2,y_2) \text{ and } (x_2,y_2)\mathcal{R}(x_3,y_3)] &\Rightarrow \left[x_1^2+y_1^2=x_2^2+y_2^2 \text{ and } x_2^2+y_2^2=x_3^2+y_3^2\right] \\ &\Rightarrow \left[x_1^2+y_1^2=x_3^2+y_3^2\right] \\ &\Rightarrow (x_1,y_1)\mathcal{R}(x_3,y_3), \end{aligned}$$

therefore $\mathcal{R}$ is a transitive relation. We deduce that $\mathcal{R}$ is an equivalence relation.

We deduce that $\mathcal{R}$ is an equivalence relation, for the equivalence class of $(1,0)$, we have

$$\begin{aligned} \mathcal{C}((1,0)) &= \left\{(x,y)\in\mathbb{R}^2 : (x,y)\,\mathcal{R}\,(1,0)\right\} \\ &= \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\right\} \end{aligned}$$

Exercise 9: (1) Let $\mathcal{R}$ be the relation defined on $\mathbb{N}^*$ by : $n\mathcal{R}m \Leftrightarrow \exists k\in\mathbb{N}^* : n=km$.
· Let $n\in\mathbb{N}^*$, we have

$$n\mathcal{R}n \Leftrightarrow \exists k\in\mathbb{N}^* : n=1m.$$

i.e. R is a reflexive relation
· Let $n,m\in\mathbb{N}^*$, we have

$$\begin{aligned} [n\mathcal{R}m \text{ and } m\mathcal{R}n] &\Leftrightarrow [(\exists k_1\in\mathbb{N}^* : n=k_1m) \text{ and } (\exists k_2\in\mathbb{N}^* : m=k_2n)] \\ &\Rightarrow (\exists k_1,k_2\in\mathbb{N}^* : n=k_1k_2n) \\ &\Rightarrow (\exists k_1,k_2\in\mathbb{N}^* : k_1k_2=1) \\ &\Rightarrow k_1=k_2=1 \\ &\Rightarrow n=m, \end{aligned}$$

so $\mathcal{R}$ is an anti-Symmetric relation.
· Let $n,m,p\in\mathbb{N}^*$, we have

$$\begin{aligned} [n\mathcal{R}m \text{ and } m\mathcal{R}p] &\Leftrightarrow [(\exists k_1\in\mathbb{N}^* : n=k_1m) \text{ and } (\exists k_2\in\mathbb{N}^* : m=k_2p)] \\ &\Rightarrow (\exists k_1,k_2\in\mathbb{N}^* : n=k_1k_2p) \\ &\Rightarrow (\exists k_3=k_1k_2\in\mathbb{N}^* : n=k_3p) \\ &\Rightarrow n\mathcal{R}p \end{aligned}$$

therefore R is a transitive relation.
(2) The relation of order $\mathcal{R}$ is partial, there exists $n=2$ and , $n=3$ such as neither $2\mathcal{R}3$, $3\mathcal{R}2$.

Exercise 10:

1. Let the function be $f:\mathbb{R}\to\mathbb{R}$, where $x\mapsto x^2$and let be.$A=[-1,4]$

   - The direct image of the set A by application f

   We have

$$\begin{aligned} f([-1,0]\cup[0,4]) &= f([-1,0])\cup f([0,4]) \\ &= [0,1]\cup[0,16]=[0,16] \end{aligned}$$

- The reciprocal (inverse) image of the set A by application f.

$x \in f^{-1}(A) \Leftrightarrow x^2 \in [-1,4]$, We exclude negative, then

$$x^2 \in [-1,4] \Rightarrow x^2 \in [0,4] \Leftrightarrow x \in [-2,2]$$

2. Let the function be $\sin : \mathbb{R} \rightarrow \mathbb{R}$.

   -The direct image of the set $\mathbb{R}$ as of $[0,2\pi] = [-1,1]$ .The direct image of the set $[0,\pi/2] = [0,1]$

   - The inverse image of set $[0,1]$:

$$\begin{aligned} f^{-1}([0,1]) &= \{x \in \mathbb{R} \ / \ \sin x \in [0,1]\} \\ &= \bigcup_{k\in\mathbb{Z}} [2k\pi, (2k+1)\pi] . \end{aligned}$$

   - The inverse image of set $[3,4]$: $f^{-1}([3,4]) = \emptyset$ (there is not a number of $\sin x \in [3,4]$)

   - The inverse image of set$[1,2]$: $f^{-1}([1,2]) = f^{-1}(\{1\}) = \left\{\frac{\pi}{2} + 2k\pi; k \in \mathbb{Z}\right\}$.

Exercise 11:
f_1 is injective, not surjective (and therefore not bijective): 1 has no precedent.
f_2 is not injective, as $(f_2(-1) = f_2(1) = 9)$ and $-1 \neq 1$. The map f_2 is not surjective, because $6 \in \mathbb{N}$ and $f_2(n) = 6 \Leftrightarrow n^2 = \frac{1}{4}$, has no solution in $\mathbb{Z}$.
f_3 is not injective, as $(f_3(-1) = f_3(1) = 1)$, nor surjective (-1 has no precedent.)
f_4 is surjective and not injective.

Exercise 12: Let the functions f and g defined from $\mathbb{N}$ to $\mathbb{N}$ by the following:

$$f(x) = 2x \text{ and } g(x) = \begin{cases} \frac{x}{2}, & \text{If } x \text{ is even} \\ 0, & \text{If } x \text{ is odd} \end{cases}$$

We have $g \circ f(x) = g(2x)$ but $2x$ is even, and therefore $g(2x) = \frac{(2x)}{2} = x$. So $g \circ f(x) = x$ On the other hand, if x is even, we have $f \circ g(x) = f\left(\frac{x}{2}\right) = x$. If x is odd $f \circ g(x) = f(0) = 0$.In particular, we have $f \circ g \neq g \circ f$ since $f \circ g(1) = 0$ while $g \circ f(1) = 1$.
f is not surjective, because odd numbers are not values taken by f. On the other hand, f is injective because if $f(x) = f(y)$, we have $2x = 2y$ and so $x = y$.
g is not injective, because $g(1) = g(3) = 0$, while $1 \neq 3$. On the other hand, g is surjective. Let us take y to be any natural integer. Then, $2y$ is even and $g(2y) = y$.
From the two previous studies, we deduce by definition that neither f nor g are bijective.

Exercise 13: Show by recurrence

1. For any $n \in \mathbb{N} : 5n^3 + n$ is divisible by 6. (Indication $n(n+1) = 2p$, $p \in \mathbb{N}$).

   Note

$$P(n) = \left(5n^3 + n\right) = 6k, \ k \in \mathbb{Z}$$

i) If $n = 0$, we have $P(0) = 0 = 6 \times 0$, so $P(0)$ is true.
ii) If $n \in \mathbb{N}$, suppose that $P(n)$ is true. We're going to show that $P(n+1)$ is true.

$$\begin{aligned} P(n+1) &= \left(5(n+1)^3 + n + 1\right) \\ &= \left(5n^3 + 15n^2 + 15n + 5 + n + 1\right) \\ &= \left(5n^3 + n + 15n^2 + 15n + 6\right) \\ &= \left(\left(5n^3 + n\right) + 15\left(n^2 + n\right) + 6\right) \\ &= (6k + 15(2p) + 6) \\ &= 6(k + 5p + 1), \quad k, p \in \mathbb{N} \\ &= 6m, \qquad m \in \mathbb{N} \end{aligned}$$

So $P(n+1)$ is true.

For any $n \in \mathbb{N} : 2^n > n$.

2. For any $n \geq 2 : 1 \cdot 2 + 2 \cdot 3 + ... + (n-1) \cdot n = \dfrac{n}{3}(n-1)(n+1)$.

Let us denote

$$P(n) : 2^n > n, \text{ for all } n \in \mathbb{N}$$

for all $n \in \mathbb{N}$. We will prove by recurrence that $P(n)$ is true for all $n \in \mathbb{N}$.

i) For $n = 0$ we have $2^0 = 1 > 0$, so $P(0)$ is true.

ii) Let $n \in \mathbb{N}$, suppose that $P(n)$ is true. We're going to show that$P(n+1)$ is true.

$$\begin{aligned} 2^{n+1} &= 2^n + 2^n \\ &> n + 2^n, \text{ because by } P(n) \text{ we know that } 2^n > n, \\ &\geq n + 1^n, \text{ because } 2^n \geq 1. \end{aligned}$$

So $P(n+1)$ is true.

3 If $x \neq 1$, $x \in \mathbb{R}$, $n \geq 1 : 1 + x + x^2 + ... + x^{n-1} = \dfrac{1 - x^n}{1 - x}$.

Let us denote

$$P(n) : 1 + x + x^2 + ... + x^{n-1} = \frac{1 - x^n}{1 - x}, \text{ for } x \in \mathbb{R} \setminus \{1\} \text{ and } n \geq 1$$

i) For $n = 1$ we have $1 = \dfrac{1 - x^1}{1 - x}$, so $P(1)$ is true.

ii) Let $n \in \mathbb{N}$, suppose that $P(n)$ is true. We're going to show that$P(n+1)$ is true.

$$\begin{aligned} P(n+1) : 1 + x + x^2 + ... + x^n &= \frac{1 - x^n}{1 - x} + x^n \\ &= \frac{1 - x^n + x^n - x^{n+1}}{1 - x} \\ &= \frac{1 - x^{n+1}}{1 - x}, \end{aligned}$$

So $P(n+1)$ is true.