Chapter 3

Real Functions of a Real Variable Dr.L. DERBAL

3.1 Introduction

In this chapter the key notion of a continuous function is introduced, followed by several important theorems about continuous functions. We deal exclusively with functions taking values in the set of real numbers (that is, real-valued functions).

3.1.1 Bounded functions, monotonic functions

Definition 3.1.1 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a function. It is said that*

a) f is said to be bounded above on D if

$$\exists M \in \mathbb{R}, \forall x \in D : f(x) \leq M.$$

b) f is said to be bounded below on D if

$$\exists m \in \mathbb{R}, \forall x \in D : f(x) \geq m.$$

c) f is bounded on D if f is both bounded above and below on D, i.e. if

$$\exists M > 0, \forall x \in D : |f(x)| \leq M \text{ or } \exists M, \exists m \in \mathbb{R}, \forall x \in D : m \leq f(x) \leq M.$$

1/ $f(x) = \cos(x)$ is bounded because $\forall x \in \mathbb{R} : -1 \leq \cos(x) \leq 1$.

2/ $f(x) = e^x$ is bounded below because $\forall x \in \mathbb{R} : e^x > 0$.

3/ $f(x) = x^2$is not bounded.

Definition 3.1.2 *Let $f : D \to \mathbb{R}$ be a function. We say that:*

a) f is increasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) \leq f(y).$$

b) f is strictly increasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) < f(y).$$

c)f is decreasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) \geq f(y).$$

d) f is strictly decreasing over D if

$$\forall x,\, y \in D,\, x < y \Rightarrow f(x) > f(y).$$

e) f is monotonic (or strictly monotonic) on D if f is increasing or decreasing (or strictly increasing or decreasing) on D.

i) The exponential function $exp : \mathbb{R} \to \mathbb{R}$ is strictly increasing.

ii) The absolute value function $x \to |x|$ defined on $\mathbb{R}$ is not monotonic.

3.1.2 Odd, even, periodic function

Definition 3.1.3 (Parity) *Let I be a symmetric interval with respect to 0 in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function. We say that:*

i) *f is even if $\forall x \in I : f(-x) = f(x)$.*

ii) *f is odd if $\forall x \in I : f(-x) = -f(x)$.*

Remark 3.1.4 *$f$is even if and only if its graph is symmetric with respect to the y-axis and f is odd if and only if its graph is symmetric with respect to at the origin.*

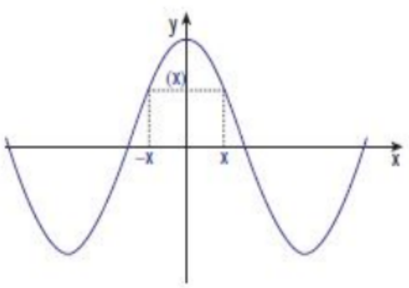

Figure 13 : Even function

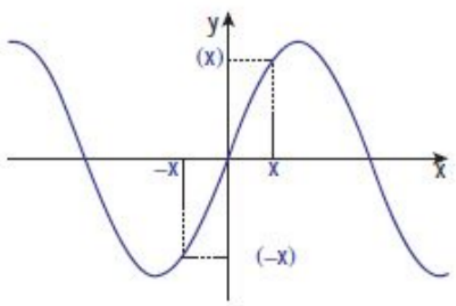

Figure 14 : Odd function

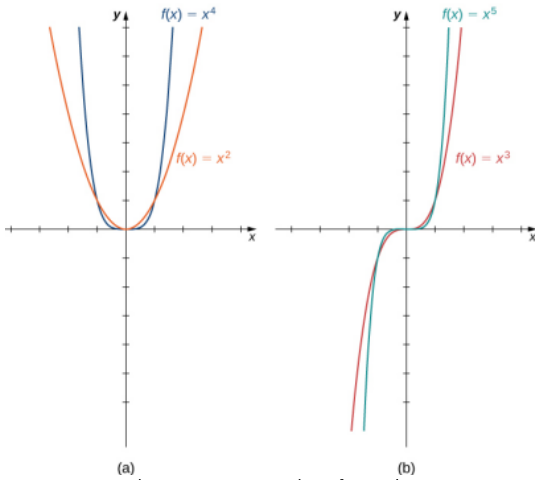

Figure 15 : Parity function

(a) For any even integer n, $f(x) = ax^n$ is an even function,

(b) For any odd integer n, $f(x) = ax^n$, is an odd function.

Definition 3.1.5 (Periodicity) *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and T be a real number, $T > 0$. The function f is called periodic of period T if $\forall x \in \mathbb{R}$, $f(x+T) = f(x)$.*

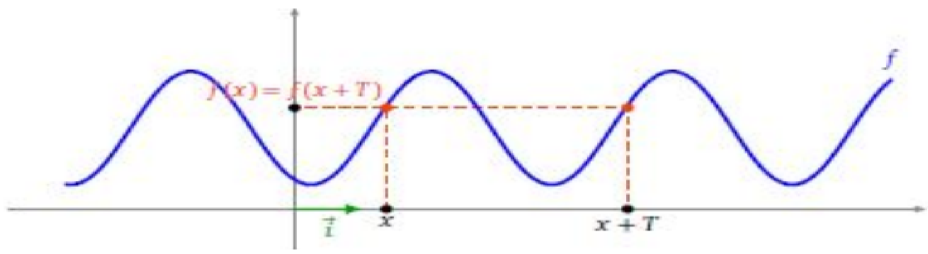

Figure 16: Periodic function

Example 3.1.6 *The functions* sin *and* cos *are 2π-periodic. The tangent function is π-periodic.*

3.1.3 Algebraic operations on functions

The set of functions of $D \subset \mathbb{R}$ in $\mathbb{R}$, is denoted $\mathcal{F}(D, \mathbb{R})$.

Definition 3.1.7 *Let f and $g \in \mathcal{F}(D, \mathbb{R})$ and $\lambda \in \mathbb{R}$. We define*

- *Sum of two functions $f + g : x \to (f+g)(x) = f(x) + g(x)$.*
- *For $\lambda \in \mathbb{R}$, $\lambda f : x \to (\lambda f)(x) = \lambda f(x)$.*
- *Product of two functions $fg : x \to (fg)(x) = f(x)g(x)$.*

Remark 3.1.8 *The functions $f+g$, λf and fg are functions belonging to $\mathcal{F}(D,\mathbb{R})$.*

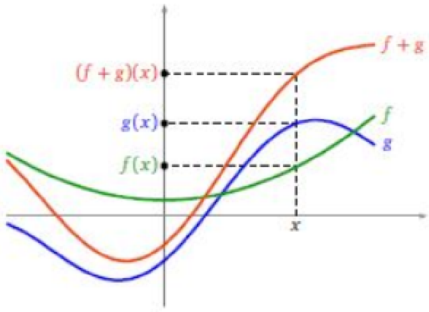

Figure 17: Sum of functions

Definition 3.1.9 *Let f and $g \in \mathcal{F}(D,\mathbb{R})$, we say that*

- $f \leq g$ *if* $\forall x \in D,\ f(x) \leq g(x)$.
- $f < g$ *if* $\forall x \in D, f(x) < g(x)$.

Example 3.1.10 *Let f and g be two functions defined on $]0,1[$ by $f(x)=x$, $g(x)=x^2$. We have $g<f$, because $\forall x \in]0,1[\ ,\ x^2 < x$.*

3.1.4 Limit of a function

General definitions

Let $f : I \to \mathbb{R}$ be a function defined over an interval I of $\mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a point of I or an end of I.

Definition 3.1.11 *Let $l \in \mathbb{R}$. We say that f has l for limit in x_0 if,*

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in I,\ |x - x_0| < \delta \Rightarrow |f(x) - l| < \epsilon.$$

In this case, we write $\lim\limits_{x \to x_0} f(x) = l$.

Example 3.1.12 *Consider the function $f(x) = 2x - 1$ which is defined on $\mathbb{R}$. At the point $x=1$, we have $\lim\limits_{x\to 1} f(x) = 1$. Indeed, for all $\epsilon > 0$, we have $|f(x)-1| = 2|x-1| < \epsilon$, if we have $|x-1| < \frac{\epsilon}{2}$. The right choice will then be to take $\delta = \frac{\epsilon}{2}$.*

Uniqueness of the limit

Proposition 3.1.13 *If f admits a limit at the point x_0, this limit is unique.*

Proof. If f admits two limits l_1 and l_2 at the point x_0, then we have, by definition, $\forall \epsilon > 0$,

$$\begin{aligned} \exists \delta_1 &> 0, \forall x \in I, \text{ if } |x - x_0| < \delta_1 \Rightarrow |f(x) - l_1| < \frac{\epsilon}{2}. \\ \exists \delta_2 &> 0, \forall x \in I, \text{ if } |x - x_0| < \delta_2 \Rightarrow |f(x) - l_2| < \frac{\epsilon}{2}. \end{aligned}$$

Let $\delta = \min(\delta_1, \delta_2) > 0$, then

$$|l_1 - l_2| \leq |f(x) - l_1| + |f(x) - l_2| \leq \epsilon$$

Since ϵ is any, for $\epsilon = \dfrac{|l_1 - l_2|}{2}$ results in $l_1 = l_2$. ■

Limit to the right, limit to the left.

Definition 3.1.14 *We say that the function f admits l as the limit to the right of x_0, as x tends to x_0^+, if for all $\epsilon > 0$ there exists a $\delta > 0$, such that: $x_0 < x < x_0 + \delta$, results in $|f(x) - l| \leq \epsilon$. In this case, we will write:*

$$\lim_{x \to x_0^+} f(x) = l \text{ or } \lim_{x \overset{>}{\to} x_0} f(x) = l.$$

We say that the function f admits l as the limit to the left of x_0, as x tends to x_0^-, if for all $\epsilon > 0$ there exists a $\delta > 0$, such that: $x_0 - \delta < x < x_0$, results in $|f(x) - l| \leq \epsilon$. In this case, we will write:

$$\lim_{x \to x_0^-} f(x) = l \text{ or } \lim_{x \overset{<}{\to} x_0} f(x) = l.$$

Example 3.1.15 *The function $x \in \mathbb{R}^+ \to \sqrt{x}$ tends to 0 when $x \to 0^+$.*

Remark 3.1.16 *If a function f has a limit l to the left of the point x_0 and a limit $l'$to the right of x_0, then the existence of a limit of x_0, is both necessary and sufficient for $l = l'$.*

Example 3.1.17 *Consider the function defined by*

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0. \end{cases}$$

It admits 1 as the limit to the right of 0 and -1 as the limit to the left of 0. But it does not admit any limit to point 0.

Cases where x becomes infinite

We will pose by definition

a) $\lim_{x\to+\infty} f(x) = l$, if

$$\forall\epsilon > 0,\ \exists A > 0, \text{ such that } x > A \Rightarrow |f(x) - l| < \epsilon.$$

b) $\lim_{x\to-\infty} f(x) = l$, if

$$\forall\epsilon > 0,\ \exists A > 0, \text{ such that } x < -A \Rightarrow |f(x) - l| < \epsilon.$$

Infinite Limit

Let $x_0 \in \mathbb{R}$. By definition, we pose

a) $\lim_{x\to x_0} f(x) = +\infty$,

$$\forall A > 0,\ \exists\delta > 0, \text{ such that } |x - x_0| < \delta \Rightarrow f(x) > A.$$

b) $\lim_{x\to x_0} f(x) = -\infty$, if

$$\forall A > 0,\ \exists\delta > 0, \text{ such that } |x - x_0| < \delta \Rightarrow f(x) < -A.$$

If $x_0 = +\infty$ or $x_0 = -\infty$, we put

a) $\lim_{x\to+\infty} f(x) = +\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x > B \Rightarrow f(x) > A.$$

b) $\lim_{x\to-\infty} f(x) = +\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x < -B \Rightarrow f(x) > A.$$

c) $\lim_{x\to+\infty} f(x) = -\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x > B \Rightarrow f(x) < -A.$$

d) $\lim_{x\to-\infty} f(x) = -\infty$,

$$\forall A > 0,\ \exists B > 0, \text{ such that } x < -B \Rightarrow f(x) < -A.$$

3.1.5 Limit theorems

Theorem 3.1.18 *Let $f : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$. The following two properties are equivalent:*

(1) $\lim\limits_{x \to x_0} f(x) = l$,

(2) For any sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in]a, b[$ such that $\lim\limits_{n \to +\infty} x_n = x_0$, then $\lim\limits_{n \to +\infty} f(x_n) = l$.

Exercise 3.1.19 *1)* $\lim\limits_{x \to 0} \sin\left(\frac{1}{x}\right)$ *does not exist, and 2)* $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Solution 3.1.20 *In fact :*

1) Let $x_n = \frac{1}{(2n+1)\frac{\pi}{2}}$, we have : $x_n \neq 0$ and $x_n \to 0$ if $n \to +\infty$. But

$$\sin\left(\frac{1}{x_n}\right) = \sin\left((2n+1)\frac{\pi}{2}\right) = (-1)^n,$$

for all $n \in \mathbb{N}$. However, this sequence does not converge (i.e. the limit does not exist).

2) Suppose $x_n \neq 0$ and $x_n \to 0$. Then

$$0 \leq \left| x_n \sin\left(\frac{1}{x_n}\right) \right| = |x_n| \left| \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|.$$

By the Gendarmes' Theorem, $\lim\limits_{n \to +\infty} \left| x_n \sin\left(\frac{1}{x_n}\right) \right| = 0$.

3.1.6 Operations of Limits

Theorem 3.1.21 *Let $f, g : [a, b] \to \mathbb{R}$ and $x_0 \in]a, b[$, such that $\lim\limits_{x \to x_0} f(x) = l$ and $\lim\limits_{x \to x_0} g(x) = l'$, Then*

a) $\lim\limits_{x \to x_0} [f(x) + g(x)] = l + l'$.

b) $\lim\limits_{x \to x_0} (\lambda f(x)) = \lambda l$ *for any* $\lambda \in \mathbb{R}$.

c) $\lim\limits_{x \to x_0} f(x) . g(x) = ll'$.

d) $\lim\limits_{x \to x_0} |f(x)| = |l|$.

e) $\lim\limits_{x \to x_0} |f(x) - l| = 0$.

f) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{l}{l'}$, *if* $l' \neq 0$.

Theorem 3.1.22 *Let $f : [a, b] \to [c, d]$ and $g : [c, d] \to \mathbb{R}$ and $x_0 \in]a, b[$, $y_0 \in [c, d]$, such that $\lim\limits_{x \to x_0} f(x) = y_0$ and $\lim\limits_{x \to y_0} g(y) = l$ Then $\lim\limits_{x \to x_0} (g \circ f)(x) = l$.*

Proposition 3.1.23 *Let* $f, g : [a, b] \to \mathbb{R}$ *and* $x_0 \in]a, b[$, *we have*

a) If $\lim_{x \to x_0} f(x) = +\infty$, *then* $\lim_{x \to x_0} \frac{1}{f(x)} = 0$.

b) If $\lim_{x \to x_0} f(x) = -\infty$, *then* $\lim_{x \to x_0} \frac{1}{f(x)} = 0$.

c) If $f \leq g$, *and* $\lim_{x \to x_0} f(x) = l$, $\lim_{x \to x_0} g(x) = l'$, *then* $l \leq l'$.

d) If $f \leq g$, *and* $\lim_{x \to x_0} f(x) = +\infty$, *then* $\lim_{x \to x_0} g(x) = +\infty$.

Theorem 3.1.24 *Let* $f, g, h : [a, b] \to \mathbb{R}$ *and* $x_0 \in]a, b[$, *we have*

i) $f(x) \leq g(x) \leq h(x)$, *for all* $x \in]a, b[$,

ii) $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l \in \mathbb{R}$.

Then $\lim_{x \to x_0} g(x) = l$.

Indeterminate forms

$+\infty - \infty$, $0 \times \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, 1^{∞}, ∞^0.

Proposition 3.1.25 *Let* f *and* g *be two functions if:*

1) f *is a bounded function in the neighbourhood of* x_0 *(*$\exists D$ *a neighbourhood of* x_0*) s.t*

$$\exists m, M \in \mathbb{R}, \forall x \in D \ : m \leq f(x) \leq M$$

2) $\lim_{x \to x_0} g(x) = 0$.

Then $\lim_{x \to x_0} f(x) \times g(x) = 0$

Example 3.1.26 *Calculate the limit* $\lim_{x \to +\infty} \frac{\sin x}{x}$.

Indeed : $\sin(\infty)$ *is not defined but it is bounded because* $|\sin x| \leq 1$ *and* $\lim_{x \to +\infty} \frac{1}{x} = 0$, *so* $\lim_{x \to +\infty} \frac{\sin x}{x} = 0$.

Equivalent functions

Definition 3.1.27 *Let* f *and* g *be two functions defined in a neighborhood of a point* x_0 ($x_0 \in \mathbb{R}$ *or* $x_0 = \pm\infty$).

We assume, moreover, that g *does not cancel in a neighborhood of* x_0, *except perhaps in* x_0 *where we can have* $g(x_0) = 0$.

We say that f *is equivalent to* g *in a neighborhood of* x_0 *if, and only if:* $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$

We denote this by $f \underset{x_0}{\sim} g$. *We also say that* f *and* g *are equivalent to the neighborhood of* x_0 *or in* x_0.

1) The functions $f(x) = \ln(x+1)$ and $g(x) = x$ are equivalent since $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$.
We note $\ln(x+1) \underset{0}{\sim} x$.

2) Always, in the vicinity of zero, $\sin x \sim x$ because $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

3.2 Continuity of a function

3.2.1 General definition

Definition 3.2.1 *Let us consider a function $f : I \to \mathbb{R}$, where I is an interval of $\mathbb{R}$. We say that f is continuous at the point $x_0 \in I$ if $\lim_{x \to x_0} f(x) = f(x_0)$, i.e. if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

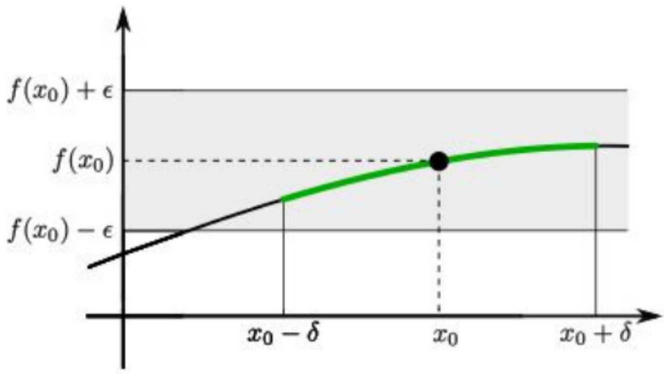

Figure 18: Continuity at the point x_0

Example 3.2.2 *Let the real function f be denied by*

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

At the point $x_0 = 0$, we have

$$|f(x) - f(x_0)| = |x \sin\left(\frac{1}{x}\right)| \leq |x|.$$

For $\epsilon > 0$, we will choose $\delta = \epsilon$. Thus

$$|x| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

So f is continuous at the point $x_0 = 0$.

Definition 3.2.3 *A function defined on an interval I is continuous on I if it is continuous at any point of I. The set of continuous functions on I is denoted by $\mathcal{C}(I)$.*

Continuity on the left, continuity on the right

Definition 3.2.4 *Let us consider a function $f : I \to R$, where I is an interval of $\mathbb{R}$.*

(1) The function f is said to be continuous on the left at x_0 if $\lim_{x \to x_0^-} f(x) = f(x_0)$, i.e.

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \in I,\ 0 < x_0 - x < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

(2) The function f is said to be continuous on the right at x_0 if $\lim_{x \to x_0^+} f(x) = f(x_0)$, i.e.

$$\forall \epsilon > 0,\ \exists \delta > 0, \forall x \in I,\ 0 < x - x_0 < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

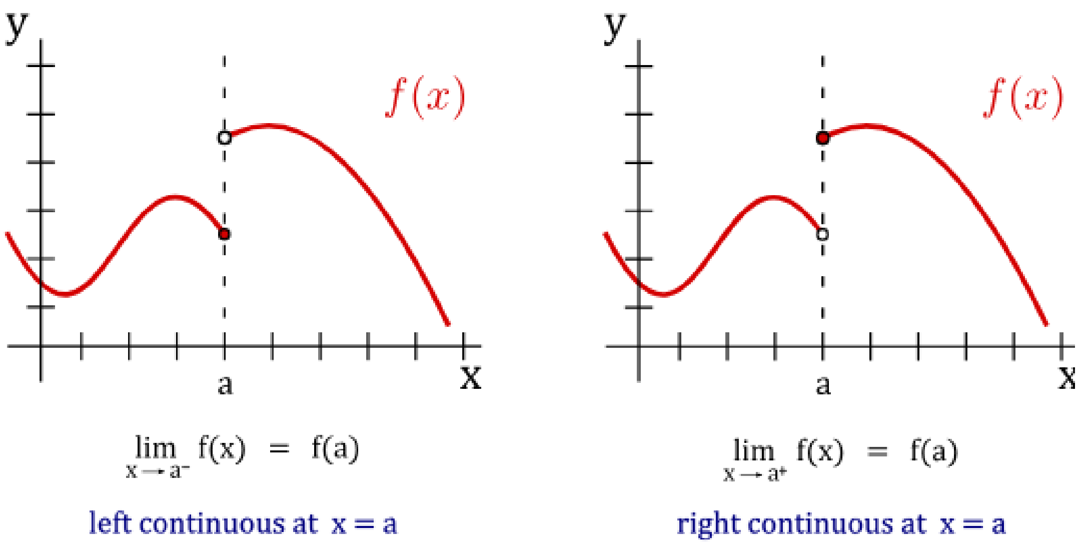

Figure 19: Left (right)Continuous at x=a

Note. The function f is continuous at x_0 if and only if f is continuous to the left and right of the point x_0. f is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$.

Summary of discontinuities

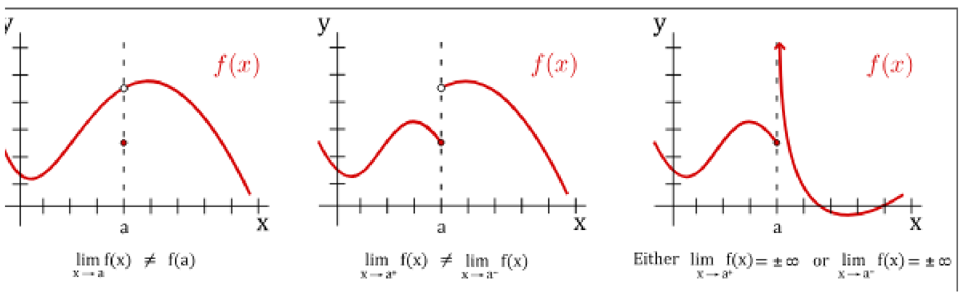

Figure 20 : discontinuity at point a

Example 3.2.5 *The function defined by*

$$f(x) = \begin{cases} 1 & \text{if } x > 0, \\ -1 & \text{if } x \leq 0. \end{cases}$$

is continuous on $\mathbb{R}^*$. *At the point* $x_0 = 0$, *the function* f *is continuous on the left, but it is not continuous on the right because*

$$\lim_{x \to 0^-} f(x) = f(0) = -1 \text{ and } \lim_{x \to 0^+} f(x) = 1 \neq f(0)$$

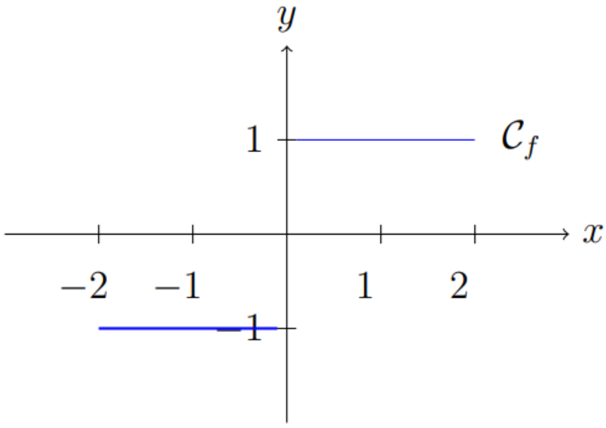

***Figure 21 : discontinuity at point* 0**

Definition 3.2.6 (Continuity on a closed interval.) *A function* f *is continuous on the closed interval* $[a, b]$ *if :*

1. *it is continuous on the open interval* (a, b);
2. *it is right continuous at point* a :

$$\lim_{x \to a^+} f(x) = f(a);$$

and

3. *it is left continuous at point* b :

$$\lim_{x \to b-} f(x) = f(b).$$

Example 3.2.7 *The function* $f(x) = \sqrt{x}$ *is continuous on the (closed) interval* $[0, +\infty)$.

The function $f(x) = \sqrt{4 - x}$ *is continuous on the (closed) interval* $(-\infty, 4]$.

Continuity extension

Definition 3.2.8 *Let I be an interval, x_0 a point of I. If the function f is not defined at the point $x_0 \in I$ and admits at this point a finite limit denoted l, the function defined by*

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \neq x_0, \\ l, & \text{if } x = x_0. \end{cases}$$

is said to be a continuity extension of f at the point x_0.

Example 3.2.9 *The function*

$$f(x) = x \sin \frac{1}{x}$$

is defined and continues on $\mathbb{R}^*$. Now, for all $x \in \mathbb{R}^*$ we have

$$|f(x)| = \left| x \sin \frac{1}{x} \right| \leq |x|$$

So $\lim\limits_{x \to 0} f(x) = 0$. The continuity extension of f to the point 0 is therefore the function $\tilde{f}$ defined by:

$$\tilde{f}(x) = \begin{cases} x \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

3.2.2 Operations on continuous functions

Definition 3.2.10 *Let I be an interval, and f and g functions defined on I and continuous at $x_0 \in I$. Then*

(1) λf is continuous at x_0, $(\lambda \in \mathbb{R})$.

(2) $f + g$ is continuous at x_0.

(3) $f.g$ is continuous at x_0.

(4) $\frac{f}{g}$ (if $g(x_0) \neq 0$) is continuous at x_0.

3.2.3 Continuity of composition function

Theorem 3.2.11 *If g is continuous at x_0 and f is continuous at $g(x_0)$, then the composition function $f \circ g$ is continuous at x_0.*

Theorem 3.2.12 *If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.*

Definition 3.2.13 (Absolute Minimum / Maximum) *Let I be an interval and $f : I \to \mathbb{R}$. Then, f achieves an absolute minimum at $c \in I$, if $\forall x \in I$, $f(x) \geq f(c)$. Similarly, f achieves an absolute maximum at $d \in I$, if $\forall x \in I$, $f(x) \leq f(d)$.*

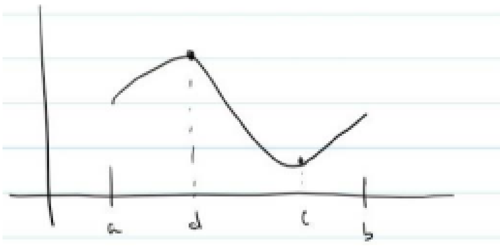

Figure 22 : Maximum and minimum

3.2.4 The Intermediate Value Theorem

Whether or not an equation has a solution is an important question in mathematics.

Theorem 3.2.14 (Intermediate Value Theorem IVT) *If f is continuous on the interval $[a,b]$ and N is between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$, then there is a number c in (a,b) such that $f(c) = N$*

The IVT guarantees that if .f is continuous and $f(a) < N < f(b)$, the line $y = N$ intersects the function at some point $x = c$. Such a number c is between a and b and has the property that $f(c) = N$ (see **Figure 23**)

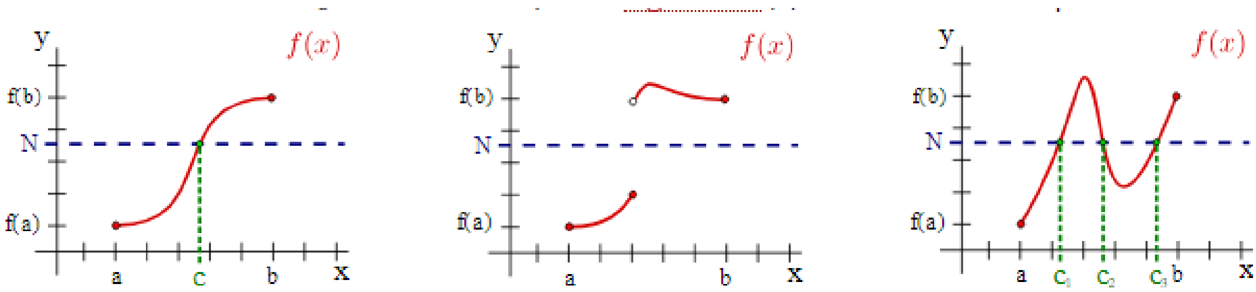

Figure 23 : Intermediate Value Theorem

(a) A continuous function where IVT holds for a single value c.
(b) A discontinuous function where IVT fails to hold.
(c) A continuous function where **IVT** holds for multiple values in (a,b).
The Intermediate Value Theorem is most frequently used for $N = 0$.

Exercise 3.2.15 *Show that there is a solution of $\sqrt[3]{x} + x = 1$ in the interval $(0,8)$.*

Solution 3.2.16 *Let $f(x) = \sqrt[3]{x} + x - 1$, $a = 0$, and $b = 8$, Since $\sqrt[3]{x}$, $x - 1$ are continuous on $\mathbb{R}$, and the sum of continuous functions is again continuous, we have that f is continuous on $\mathbb{R}$, thus in particular, f is continuous on $[0, 8]$. We have $f(a) = f(0) = \sqrt[3]{0} + 0 - 1 = -1$ and $f(b) = f(8) = \sqrt[3]{8} + 8 - 1 = 9$. Thus $N = 0$ lies between $f(a) = -1$ and $f(b) = 9$, so the conditions of the* ***IVT*** *are satisfied. So, there exists a number c in $(0, 8)$ such that $f(c) = 0$. This means that c satisfies$\sqrt[3]{c} + c - 1 = 0$, in otherwords, is a solution for the equation given.*

Alternatively we can let $f(x) = \sqrt[3]{x} + x$, $N = 1$, $a = 0$ and $b = 8$. Then as before f is the sum of two continuous functions, so is also continuous everywhere, in particular, continuous on the interval $[0, 8]$, $f(a) = f(0) = \sqrt[3]{0} + 0 = 0$ and $f(b) = f(8) = \sqrt[3]{8} + 8 = 10$. Thus $N = 1$ lies between $f(a) = 0$ and $f(b) = 10$, so the conditions of the ***IVT*** *are satisfied. So, there exists a number c in $(0, 8)$ such that $f(c) = 1$. This means that c satisfies $\sqrt[3]{c} + c = 1$, in otherwords, is a solution for the equation given.*

Proposition 3.2.17 *Let f be a continuous function on interval $[a, b]$, such that $f(a).f(b) < 0$, there exists $c \in]a, b[$ such that $f(c) = 0$.*

3.2.5 Uniform Continuity

Recall the definition of continuityon an interval I : let $x_0 \in I$ we have

$$\forall \epsilon > 0, \exists \delta(\epsilon, x_0) > 0, \forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon.$$

Here, $\delta(\epsilon, x_0)$ indicates that δ can depend on ϵ and x_0.

Exercise 3.2.18 *Consider the function $f(x) = \dfrac{1}{x}$. f is continuous on $(0, 1)$.*

Solution 3.2.19 *We want to show that if $|x - x_0| < \delta$, then $|\dfrac{1}{x} - \dfrac{1}{x_0}| < \epsilon$. Specifically, we can choose $\delta = \min\left\{\dfrac{x_0}{2}, \dfrac{x_0^2}{2}\epsilon\right\}$.*

In fact

$$|x - x_0| < \frac{x_0}{2} \Rightarrow |x| > x_0 - |x - x_0| > \frac{x_0}{2}.$$

Thus, $\frac{1}{|x|} < \frac{2}{x_0}$. *Therefore,*

$$\begin{aligned} |\frac{1}{x} - \frac{1}{x_0}| &= \frac{|x - x_0|}{|xx_0|} \\ &< \frac{\delta}{|x||x_0|} \\ &< \frac{2\delta}{x_0^2} \\ &< \frac{2\frac{x_0^2}{2}\epsilon}{x_0^2} = \epsilon. \end{aligned}$$

Definition 3.2.20 (Uniformly Continuous) *Let* $f : I \to \mathbb{R}$. *Then* f *is uniformly continuous on* I *if*

$$\forall \epsilon > 0, \exists \delta(\epsilon) > 0, \forall x, y \in I, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon.$$

Remark 3.2.21 *Thus, in the definition of uniform continuity,* δ *only depends on* ϵ*!.*

Example 3.2.22 *The function* $f(x) = x^2$ *is uniformly continuous on* $[0, 1]$.
Let $\epsilon > 0$. *Choose* $\delta = \frac{\epsilon}{2}$. *Then, if* $x, y \in [0, 1]$ *then* $|x - y| < \delta$ *implies that*

$$|x^2 - y^2| = |x - y||x + y| \leq 2|x - y| < 2\delta = \epsilon.$$

Remark 3.2.23 *There are continuous functions that are not uniformly continuous.*

Example 3.2.24 $f(x) = \frac{1}{x}$ *is not uniformly continuous on* $(0, 1)$*, but first we consider the negation of the definition.*

Negation of uniform continuity : f *is not uniformly continuous on* I *if*

$$\exists \epsilon_0 > 0, \forall \delta > 0, \exists x, y \in I, |x - y| < \delta \Rightarrow |f(x) - f(y)| > \epsilon_0.$$

Let $\delta > 0$*, choose* $\epsilon_0 = 2$*,* $y = \min\left\{\delta, \frac{1}{2}\right\}$ *and* $x = \frac{y}{2}$. *Then* $|x - y| = \frac{y}{2} \leq \frac{\delta}{2} < \delta$ *and*

$$|\frac{1}{x} - \frac{1}{y}| = |\frac{2}{y} - \frac{1}{y}| = \frac{1}{y} \geq 2$$

Theorem 3.2.25 *Let* $f : [a, b] \to \mathbb{R}$. *Then,* f *is continuous if and only if* f *is uniformly continuous.*

The following procedure is a practical method of showing that a function is uniformly continuous.

Definition 3.2.26 *A function f definite of $I \subset \mathbb{R}$ in $\mathbb{R}$ is said to be k-Lipschitzian over I if:*

$$\exists k \geq 0, \forall x, y \in I : |f(x) - f(y)| \leq k|x - y|$$

Remark 3.2.27 *A k-Lipschitzian function on I is uniformly continuous on I.*

Indeed; for $\epsilon > 0$, we just need to take $\delta = \dfrac{\epsilon}{k}$, such that

$$\forall x, y \in I : |x - y| < \delta \Rightarrow |f(x) - f(y)| \leq k|x - y| < \epsilon.$$

Definition 3.2.28 *A function f is said to be contracting on I if f is k-Lipschitzian with $0 \leq k < 1$.*

Conclusion 3.2.29 *A contracting function on I is uniformly continuous on I*

Here is a theorem very used in practice to show that a function is bijective.

Theorem 3.2.30 *Let $f : I \rightarrow \mathbb{R}$ be a function defined on an interval I of $\mathbb{R}$. If f is continuous and strictly monotonic on I, so*

1. f establishes a bijection of the interval I in the image interval $J = f(I)$,
2. The inverse function $f^{-1} : J \rightarrow I$ is continuous and strictly monotonic on J and it has the same direction of variation as f.

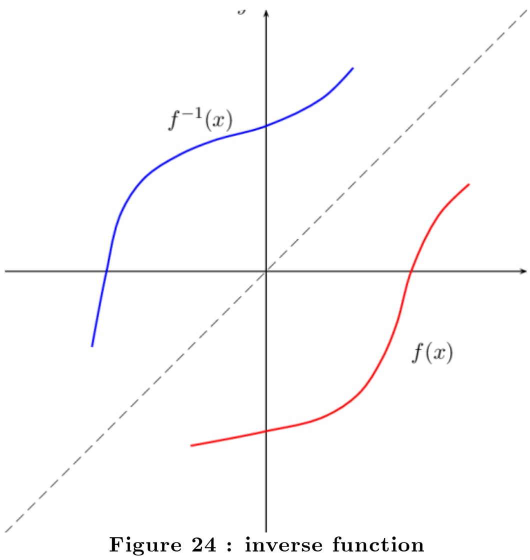

Figure 24 : inverse function

3.3 Derivable function

3.3.1 Definition and properties

Definition 3.3.1 *Let f be defined in a δ-neighbourhood $(x_0-\delta, x_0+\delta)$ of $x_0 \in \mathbb{R}$ $(\delta > 0)$. We say that f is **differentiable** at x_0 if the limit*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

*exists in $\mathbb{R}$. This limit, denoted by $f'(x_0)$, is called the **derivative** of f at x_0 .*

Furthermore, if f is differentiable at every $x_0 \in I$ (an interval), we write f' or $\frac{df}{dx}$ for the function f'.

Example 3.3.2 *1)* $f(x) = c \Rightarrow f'(x_0) = \lim_{x\to x_0} \frac{c-c}{x-x_0} = 0 \Rightarrow f'(x) = 0,\ \forall x \in \mathbb{R}.$

2) $f(x) = x^2 \Rightarrow f'(x_0) = \lim_{x\to x_0} \frac{x^2-x_0^2}{x-x_0} = \lim_{x\to x_0}(x+x_0) = 2x_0 \Rightarrow f'(x) = 2x.$

3) $f(x) = \sqrt{x} \Rightarrow f'(x_0) = \lim_{x\to x_0} \frac{\sqrt{x}-\sqrt{x_0}}{x-x_0} = \lim_{x\to x_0} \frac{1}{\sqrt{x}+\sqrt{x_0}} = \frac{1}{2\sqrt{x_0}} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}.$

Remark 3.3.3 *By substituting $x - x_0 = h$, we find:*

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \text{ exists and is finite} \Leftrightarrow (\ f \text{ is derivative at } x_0)$$

Example 3.3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. The derivative of f at a point $x_0 \in \mathbb{R}$ is*

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{(x_0+h)^2-x_0^2}{h} \\ &= \lim_{h\to 0} \frac{h^2+2hx_0}{h} = \lim_{h\to 0}(h+2x_0) = 2x_0 \end{aligned}$$

Theorem 3.3.5 *If $f : I \to \mathbb{R}$ is differentiable at $x_0 \in I$, then f is continuous at x_0.*

Proof.

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

Hence

$$\begin{aligned} \lim_{h\to 0}(f(x)-f(x_0)) &= \lim_{h\to 0}(f(x_0+h)-f(x_0)) \\ &= \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} h \\ &= f'(x_0) \lim_{h\to 0} h = 0. \end{aligned}$$

Therefore

$$\lim_{h \to 0} f(x) = f(x_0)$$

■

Example 3.3.6 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$. Then f is differentiable at any $x \in \mathbb{R} - \{0\}$. But f is not differentiable at 0.*

In fact, we have :

If $x > 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 1.$$

If $x < 0$, then

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = -1.$$

Therefore, the derivative does not exist at 0, as

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} \neq \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

Note that the function f in the above example is continuous at 0 : thus, continuity does not imply differentiability. However, the converse is true.

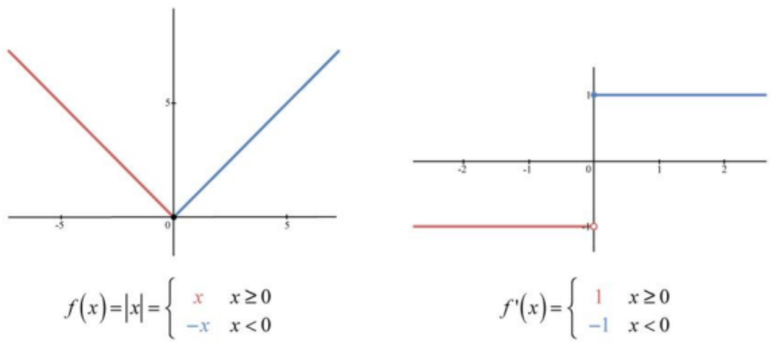

Figure 25 : f and f' s.t $f(x)=|x|$

3.3.2 One-sided derivatives

1) In a manner similar to the definition of the one-sided limit, we may also define the **left** and **right derivatives** of f at x_0 via

$$f'_-(x_0) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}, \quad f'_+(x_0) = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

2) $$\begin{pmatrix} f \text{ is derivative on the right and left at } x_0 \\ \text{and} \\ f'_-(x_0) = f'_+(x_0) \end{pmatrix} \Leftrightarrow \begin{pmatrix} f \text{ is derivative at } x_0 \\ \text{and} \\ f'(x_0) = f'_-(x_0) = f'_+(x_0) \end{pmatrix}$$

3) If $f'_{-}(x_0) \neq f'_{+}(x_0)$, then f is not differentiable at x_0 and we say that x_0 is an angular point.

Remark 3.3.7 *If f is differentiable at $x_0 \in \mathbb{R}$ then there exists a function $\varepsilon(x)$ such that $\lim\limits_{x\to 0}\varepsilon(x) = 0$ and*

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0)$$

Indeed, define

$$\varepsilon(x) := \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)$$

Then $\varepsilon(x) \to 0$ as $x \to 0$ and $f(x) = f(x_0) + f'(x_0)(x - x_0) + \varepsilon(x)(x - x_0)$

This enables one to re-interpret the formula in the above Remark as follows. If f is differentiable at $x_0 \in \mathbb{R}$, then one can write for the value of $f(x = x_0 + h)$, that is "near" x_0:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + o(h)$$

where the notation $o(h)$ reads as "little o of h", and denotes any function which has the following property: $\lim\limits_{h\to 0}\dfrac{o(h)}{h} = 0$.

3.3.3 Operations on derivative functions:

Theorem 3.3.8 *Let $f : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. Then,*

1. *(Linearity) $\forall\alpha \in \mathbb{R}$, $(\alpha f + g)'(x_0) = \alpha f'(x_0) + g'(x_0)$.*
2. *(Product rule) $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.*
3. *(Quotient rule) If $g(x) \neq 0$ for all $x \in I$, then $\left(\dfrac{f}{g}\right)'(x_0) = \dfrac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$.*

Proof.

1.

$$\begin{aligned}\lim_{x\to x_0}\frac{(\alpha f + g)(x) - (\alpha f + g)(x_0)}{x - x_0} &= \lim_{x\to x_0}\left(\alpha\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0}\right) \\ &= \alpha f'(x) + g'(x).\end{aligned}$$

2. We first write

$$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0}g(x) + f(x_0)\frac{g(x) - g(x_0)}{x - x_0}$$

then

$$\begin{aligned}\lim_{x\to x_0}\frac{(fg)(x)-(fg)(x_0)}{x-x_0} &= \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{x-x_0}g(x)+f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right]\\ &= f'(x_0)g(x_0)+f(x_0)g'(x_0)\end{aligned}$$

3.The result follows from

$$\begin{aligned}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(x_0)}{x-x_0} &= \frac{f(x)g(x_0)-g(x)f(x_0)}{g(x)g(x_0)(x-x_0)}\\ &= \frac{f(x)-f(x_0)}{g(x)g(x_0)(x-x_0)}g(x_0)-\frac{g(x)-g(x_0)}{g(x)g(x_0)(x-x_0)}f(x_0)\end{aligned}$$

then

$$\begin{aligned}\lim_{x\to x_0}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(x_0)}{x-x_0} &= \lim_{x\to x_0}\frac{f(x)g(x_0)-g(x)f(x_0)}{g(x)g(x_0)(x-x_0)}\\ &= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{g(x)g(x_0)(x-x_0)}g(x_0)-\frac{g(x)-g(x_0)}{g(x)g(x_0)(x-x_0)}f(x_0)\right)\\ &= \frac{f\prime(x_0)g(x_0)-f(x_0)g\prime(x_0)}{g^2(x_0)}.\end{aligned}$$

■

Theorem 3.3.9 *If g is differentiable at $x_0\in\mathbb{R}$ and f is differentiable at $g(x_0)$, then $f\circ g$ is differentiable at x_0 and*

$$(f\circ g)'(x_0)=f'(g(x_0))\,g'(x_0)$$

Proof. By definition of the derivative and Remark 4.3.7, we have

$$f(y)-f(y_0)=f'(y_0)(y-y_0)+\varepsilon(y)(y-y_0)$$

where $\varepsilon(y)\to 0$ as $y\to y_0$. substitute y and y_0 in the above equality by $y=g(x)$ and $y_0=g(x_0)$, then divide both sides by $x-x_0$, to get

$$\frac{f(g(x))-f(g(x_0))}{x-x_0}=f'(g(x_0))\frac{g(x)-g(x_0)}{x-x_0}+\varepsilon(g(x))\frac{g(x)-g(x_0)}{x-x_0}.$$

By Theorem 4.3.5, g is continuous at x_0. Hence $y=g(x)\to g(x_0)=y_0$ as $x\to x_0$, and $\varepsilon(g(x))\to 0$ as $x\to x_0$.Passing to limit $x\to x_0$ in the above equality yields the required result. ■

Theorem 3.3.10 *Let f be continuous and strictly increasing on (a, b). Suppose that, for some $x_0 \in (a, b)$, f is differentiable at x_0 and $f'(x_0) \neq 0$. Then the inverse function $g = f^{-1}$is differentiable at $y_0 = f(x_0)$ and*

$$g'(y_0) = \frac{1}{f'(x_0)} \quad \textit{(writes } x_0 \textit{ as a function of } y_0\textit{)}$$

Example 3.3.11 *Define*

$$\begin{array}{rl} f: & \mathbb{R} \to \mathbb{R} \\ & x \mapsto x^2 \end{array}.$$

Then $f'(0) = 0$ and f is not invertible on any neighborhood of the origin, because it is non-monotonic. On the other hand, if

$$\begin{array}{rl} f: &]0, +\infty[\to]0, +\infty[\\ & x \mapsto x^2 \end{array},$$

then $f\prime(x) = 2x \neq 0$ and the inverse function f^{-1} : is given by

$$f^{-1}(y) = \sqrt{y}.$$

The formula for the inverse of the derivative gives

$$\left(f^{-1}\right)'\left(x^2\right) = \frac{1}{f'(x)} = \frac{1}{2x}$$

or, writing $x = f^{-1}(y)$,

$$\left(f^{-1}\right)'(y) = \frac{1}{2\sqrt{y}}.$$

Example 3.3.12 *Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Then f is strictly increasing. The inverse function $f^{-1} : \mathbb{R} \to \mathbb{R}$ is given by*

$$f^{-1}(y) = y^{\frac{1}{3}}.$$

Then $f'(0) = 0$ and f^{-1} is not differentiable at $f(0) = 0$, On the other hand, $f^{-1}$is differentiable at non-zero points of $\mathbb{R}$, with

$$\left(f^{-1}\right)'\left(x^3\right) = \frac{1}{f'(x)} = \frac{1}{3x^2}$$

or, writing $x = y^{\frac{1}{3}}$,

$$\left(f^{-1}\right)'(y) = \frac{1}{3y^{\frac{2}{3}}}.$$

3.3.4 Derivative of usual functions

μ represents a function $x \to \mu(x)$.

function	derivative
x^n	nx^{n-1} $(n \in \mathbb{Z})$
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\sqrt{x}$	$\frac{1}{2}\frac{1}{\sqrt{x}}$
x^α	$\alpha x^{\alpha-1}$ $(\alpha \in \mathbb{R})$
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1+\tan^2 x = \frac{1}{\cos^2 x}$

function	derivative
μ^n	$n\mu'\mu^{n-1}$, $(n \in \mathbb{Z})$
$\frac{1}{\mu}$	$-\frac{\mu'}{\mu^2}$
$\sqrt{\mu}$	$\frac{1}{2}\frac{\mu'}{\sqrt{\mu}}$
μ^α	$\alpha\mu'\mu^{\alpha-1}$, $(\alpha \in \mathbb{R})$
e^μ	$\mu' e^\mu$
$\ln \mu$	$\frac{\mu'}{\mu}$
$\cos \mu$	$-\mu' \sin \mu$
$\sin \mu$	$\mu' \cos \mu$
$\tan \mu$	$\left(1+\tan^2 \mu\right)\mu' = \frac{\mu'}{\cos^2 \mu}$

3.3.5 The n^{th} derivative

Definition 3.3.13 *Let $f : I \to \mathbb{R}$ be a differentiable function and let f' be its derivative. If the function $f' : I \to \mathbb{R}$ is also differentiable, we denote $f'' = (f')'$ the second derivative of f. More generally we note:*

$$f^{(0)} = f,\ f^{(1)} = f',\ f^{(2)} = f'' \text{ and } f^{(n+1)} = \left(f^{(n)}\right)'$$

If the n^{th} derivative $f^{(n)}$ exists, we say that f is n times differentiable.

- *If f is n times differentiable on I and $f^{(n)}$ is continuous on I, we say that f belongs to class C^n, and we write $f \in C^n(I, \mathbb{R})$.*

- *If f is differentiable an infinite number of times, i.e., $\forall n \in \mathbb{N}$, $f^{(n)}$ exists and is continuous, we say that f belongs to class C^∞, and we write $f \in C^\infty(I, \mathbb{R})$*

- *If f is continuous but not differentiable, we say that f belongs to class C^0, and we write $f \in C^0(I, \mathbb{R})$.*

Example 3.3.14 *Polynomial functions, $\cos x$, $\sin x$, e^x are functions belonging to class C^∞ on $\mathbb{R}$.*

Example 3.3.15 *Computing the n^{th} derivative of the function $f(x) = \ln x$*

$$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x},\ f''(x) = \frac{(-1).1}{x^2},\ f^{(3)}(x) = \frac{(-1)^2.1.2}{x^3}$$

and from this

$$f^{(n)}(x) = \frac{(-1)^{n-1}.(n-1)!}{x^n}.\ \forall n \geq 1.$$

And the proof by induction.

Example 3.3.16 *Using the same method, prove that.*

$$\begin{aligned} \sin^{(n)} x &= \sin\left(x + n\frac{\pi}{2}\right) \\ \cos^{(n)} x &= \cos\left(x + n\frac{\pi}{2}\right) \end{aligned}$$

For $\sin^{(n)} x$*: we have*

$$\begin{aligned} \sin x^{(1)} &= \cos x = \sin\left(x + \frac{\pi}{2}\right) \\ \sin^{(2)} x &= \sin\left(x + \frac{\pi}{2}\right)' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2\frac{\pi}{2}\right) \\ \sin^{(3)} x &= \sin\left(x + 2\frac{\pi}{2}\right)' = \cos\left(x + 2\frac{\pi}{2}\right) = \sin\left(x + 3\frac{\pi}{2}\right) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sin^{(n)} x &= \sin\left(x + n\frac{\pi}{2}\right) \end{aligned}$$

In the same way we demonstrate the second.

Leibniz's rule:

Let f and g be two functions belonging to class $C^n(I, \mathbb{R})$. Then $f.g$ is also a function in class $C^n(I, \mathbb{R})$, and we have:

$$(f.g)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)},$$

where $C_n^k = \dfrac{n!}{(n-k)!k!}$.

Proof of Leibniz Rule

The Leibniz rule can be proved with the help of mathematical induction. Let $f(x)$ and $g(x)$ be n times differentiable functions. Applying the initial case of mathematical induction for $n = 1$ we have the following expression.

$$(f(x).g(x))' = f'(x).g(x) + f(x).g'(x)$$

Which is the simple product rule and it holds true for $n = 1$. Let us assume that this statement is true for all $n > 1$, and we have the below expression.

$$\begin{aligned}
(f.g)^{(n)} &= \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)} = f^{(0)} g^{(n)} + ... + C_n^k f^{(k)} g^{(n-k)} + ... + f^{(n)} g^{(0)} \\
(f.g)^{(n+1)} &= \left((f.g)^{(n)} \right)' = \sum_{k=0}^{n} C_n^k \left(f^{(k)} g^{(n-k)} \right)' \\
&= \sum_{k=0}^{n} C_n^k \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n+1-k)} \right) \\
&= \sum_{k=0}^{n} C_n^k f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n+1-k)}.
\end{aligned}$$

We change the variable in the first sum: $p = k + 1$

$$\sum_{k=0}^{n} C_n^k f^{(k+1)} g^{(n-k)} = \sum_{p=1}^{n+1} C_n^{p-1} f^{(p)} g^{(n+1-p)}.$$

Therefore:

$$(f.g)^{(n+1)} = \sum_{k=1}^{n+1} C_n^{k-1} f^{(k)} g^{(n+1-k)} + \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n+1-k)},$$

consequently

$$(f.g)^{(n+1)} = \left(\sum_{k=1}^{n} \left(C_n^{k-1} + C_n^k \right) f^{(k)} g^{(n+1-k)} \right) + C_n^n f^{(n+1)} g^{(0)} + C_n^0 f^{(0)} g^{(n+1)}.$$

Note that $C_n^n = C_n^0 = 1$ and $C_n^{k-1} + C_n^k = C_{n+1}^k$ then

$$(f.g)^{(n+1)} = \left(\sum_{k=1}^{n} C_{n+1}^k f^{(k)} g^{(n+1-k)} \right) + f^{(n+1)} g^{(0)} + f^{(0)} g^{(n+1)}.$$

Note that we can include the last two terms in the sum

$$\begin{aligned}
&C_{n+1}^0 f^{(0)} g^{(n+1)} = f^{(0)} g^{(n+1)} \quad \text{and} \\
&C_{n+1}^{n+1} f^{(n+1)} g^{(n+1-n+1)} = f^{(n+1)} g^{(0)},
\end{aligned}$$

then

$$(f.g)^{(n+1)} = \sum_{k=0}^{n+1} C_{n+1}^k f^{(k)} g^{(n+1-k)}.$$

So, according to the proof by induction

$$(\forall n \in \mathbb{N}) \, (\forall x \in I) : (f.g)^{(n)} (x) = \sum_{k=0}^{n} C_n^k f^{(k)} (x) \, g^{(n-k)} (x) \, .$$

Example 3.3.17 *Calculate the n^{th} derivative of the function:*

$$f(x) = \left(x^2 + x\right) \ln x \ \textit{and} \quad g(x) = e^x \sin x.$$

1) $f(x) = \left(x^2 + x\right) \ln x = f_1(x) . g_1(x)$ *or* $f_1(x) = x^2 + x$ *and* $g_1(x) = \ln x$
$f_1^{(0)}(x) = x^2 + x \Rightarrow f_1^{(1)}(x) = 2x + 1,\ f_1^{(2)}(x) = 2,\ f_1^{(k)}(x) = 0$ *for all* $k \geq 3\,(k \in \mathbb{N})$,
$g_1^{(n)}(x) = \dfrac{(-1)^{n-1} . (n-1)!}{x^n}$,

$$\begin{aligned} f^{(n)}(x) &= (f_1.g_1)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f_1^{(k)} g_1^{(n-k)}(x) \\ &= C_n^0 f_1^{(0)} g_1^{(n)}(x) + C_n^1 f_1^{(1)} g_1^{(n-1)}(x) + C_n^2 f_1^{(2)} g_1^{(n-2)}(x) + 0 \\ &= \left(x^2 + x\right) \frac{(-1)^{n-1} . (n-1)!}{x^n} + n(2x+1) \frac{(-1)^{n-2} . (n-2)!}{x^{n-1}} + \frac{n(n-1)}{2} .2. \frac{(-1)^{n-3} . (n-3)!}{x^{n-2}}. \end{aligned}$$

2) $g(x) = e^x \sin x = f_2(x) . g_2(x)$ *or* $f_2(x) = e^x$ *and* $g_2(x) = \sin x$.
$f_2^{(n)}(x) = e^x$ *and* $g_2^{(n)}(x) = \sin\left(x + n\frac{\pi}{2}\right)$

$$\begin{aligned} g^{(n)}(x) &= (f_2.g_2)^{(n)}(x) = \sum_{k=0}^{n} C_n^k f_2^{(k)} g_2^{(n-k)}(x) \\ &= \sum_{k=0}^{n} C_n^k e^x \sin\left(x + (n-k)\frac{\pi}{2}\right). \end{aligned}$$

Definition 3.3.18 (Critical Points) *Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = O$ or $f'(c)$ is undefined.*

Theorem 3.3.19 (Fermat's Theorem) *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.*

Exercise 3.3.20 *Find the local extremum (maximum and minimum) over the specified interval*

$$f(x) = -x^2 + 3x - 2 \quad \textit{over } [1, 3].$$

Solution 3.3.21 ***Step 1.*** *Evaluate f at the endpoints $x = 1$ and $x = 3$.*

$$f(1) = 0 \ \textit{and}\ f(3) = -2.$$

Step 2. *Since $f'(x) = -2x + 3 = 0$ at $x = \frac{3}{2}$ and $\frac{3}{2}$ is in the interval $[1, 3]$, $f\left(\frac{3}{2}\right) = \frac{1}{4}$ is a candidate for a local extremum of f over $[1, 3]$.*

Step 3. *We compare the values found in steps 1 and 2.We find that the local extremum minimum of f is -2, and it occurs at $x = 3$. The local extremum maximum of f is $\frac{1}{4}$, and it occurs at $x = \frac{3}{2}$ as shown in Figure*

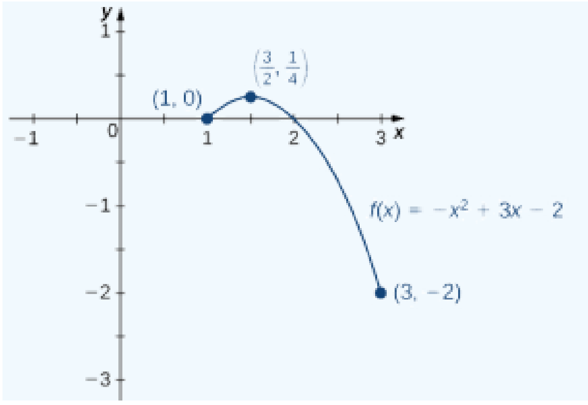

Figure 26 : This function has both local extremum maximum and minimum.

Method of finding points where the function f possesses extreme values:

Theorem 3.3.22 *Let $f \in F(D, \mathbb{R})$ be differentiable on D, assuming that f'' exists,let $x_0 \in D$ then:*

$$\begin{cases} f'(x_0) = 0 \\ f''(x_0) > 0 \end{cases} \Longrightarrow x_0 \text{ is a local minimum point of } f$$

$$\begin{cases} f'(x_0) = 0 \\ f''(x_0) < 0 \end{cases} \Longrightarrow x_0 \text{ is a local minimum point of } f$$

Example 3.3.23 *Let the function $f(x) = \cos x$ and $x_0 = 0$, $x_1 = \pi$.*

$$f'(x) = -\sin x \Rightarrow \begin{cases} f'(0) = 0 \\ f'(\pi) = 0 \end{cases} \Rightarrow x_0 \text{ and } x_1 \text{ are critical points.}$$

$$f''(x) = -\cos x \Rightarrow \begin{cases} f''(0) = -1 < 0 \to x_0 = 0 \text{ is a local maximum point of } f. \\ f''(\pi) = 1 > 0 \ \to x_1 = \pi \text{ is a local minimum point of } f. \end{cases}.$$

The general case: Let $f \in C^{(n)}(D, \mathbb{R})$, where:

$$f'(x_0) = f''(x_0) = ... = f^{(n-1)}(x_0) = 0, \ f^{(n)}(x_0) \neq 0$$

Case1: If n is even

$$\begin{aligned} f^{(n)}(x_0) &> 0 \Rightarrow x_0 \text{ is a local minimum point of } f. \\ f^{(n)}(x_0) &< 0 \Rightarrow x_0 \text{ is a local maximum point of } f \end{aligned}$$

Case2: If n is odd

$$f^{(n)}(x_0) \neq 0 \Rightarrow x_0 \text{ is not extreme point but rather an inflection point.}$$

Example 3.3.24 *$x_0 = 0$ and $f(x) = x^3$*

$$\begin{aligned} f'(x) &= 3x^2 \Rightarrow f'(0) = 0 \to f \text{ has a critical point at } x_0 = 0 \\ f''(x) &= 6x \Rightarrow f''(0) = 0 \\ f'''(x) &= 6 \Rightarrow f'''(0) \neq 0 \end{aligned}$$

With $n = 3$ being an odd number and $f'''(x) \neq 0$, hence $x_0 = 0$ is an inflection point, and f does not possess an extreme value at $x_0 = 0$.

Example 3.3.25 *Let the function $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 6\ln x - 2x^3 + 9x^2 - 18x$. Does f have an extreme value at $x_0 = 0$?*

$$\begin{aligned} f'(x) &= \frac{6}{x} - 6x^2 + 18x - 18 \Rightarrow f'(1) = 0 \\ f''(x) &= -\frac{6}{x^2} - 12x + 18 \Rightarrow f''(1) = 0 \\ f'''(x) &= \frac{12}{x^3} - 12 \Rightarrow f'''(1) = 0 \\ f^{(4)}(x) &= -\frac{36}{x^4} \Rightarrow f^{(4)}(1) \neq 0. \end{aligned}$$

Since $n = 4$ is even number and $f^{(4)}(1) < 0$, then $x_0 = 1$is a local maximum point of f and $f(1) = -11$ is the local maximum value of f.

Theorem 3.3.26 (Rolle's Theorem) *Let $f : [a, b] \to \mathbb{R}$ be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval $]a, b[$ such that*

$$f(a) = f(b),$$

then there exists at least one $c \in]a, b[$ such that $f'c) = 0$.

Proof. - If f is constant over $[a, b]$ then it is obvious ($f' = 0$).

-Otherwise; since f is continuous on $[a, b]$ then it is bounded on $[a, b]$, so

$$\sup_{x \in]a,b[} f(x) = M,$$

exists, then we have

$$\forall x \in]a, b[: f(x) \leq M,$$

we can assume that M is different from $f(a) = f(b)$ and therefore there exists c in $]a,b[$ such that $M = f(c)$, therefore

$$\forall x \in]a,b[: f(x) \leq f(c),$$

then c is a local maximum of f so according to Fermat's theorem $f'(c) = 0$. ■

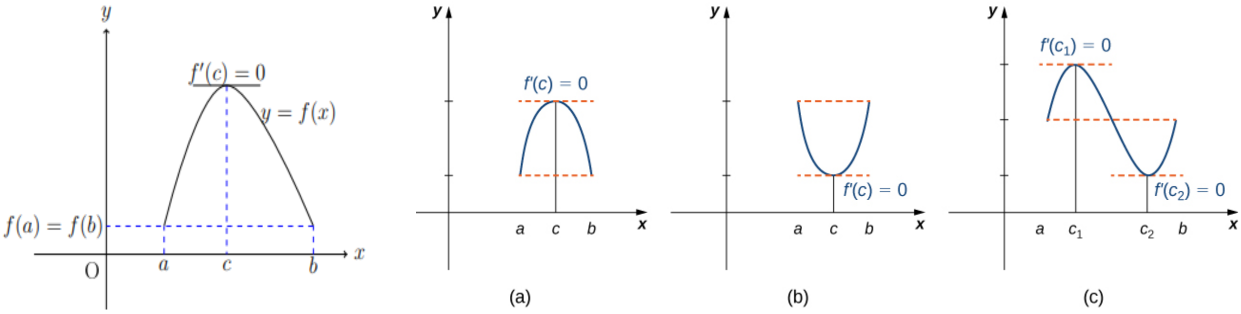

Figure 27 :

- If a differentiable function f satisfies $f(a) = f(b)$, then its derivative must be zero at some point (s) between a and b
- This means that the curve at the point $(c, f(c))$ accepts a tangent parallel to the x-axis.

Example 3.3.27 *Can Rolle's Theorem be applied to the function* $f(x) = x^2 + 1$ *in the interval* $[-1,1]$*?*

We have f *is continuous in the interval* $[-1,1]$*, differentiable on* $]-1,1[$*, and* $f(1) = f(-1)$*. Therefore, Rolle's Theorem can be applied.*

Example 3.3.28 *Can Rolle's Theorem be applied to the function* $f(x) = (|x| - 1)^2$ *on* $[-1,1]$.

We have f *is continuous over* $[-1,1]$ *and* $f(1) = f(-1) = 0$*, but* $f'(c) \neq 0$ *for any* $c \in]-1,1[$*because* f *is not differentiable at* $x = 0$*, the conditions of Rolle's theorem are not satisfied. In fact, the conclusion does not hold here; there is no* $c \in]-1,1[$*, such that* $f'(c) \neq 0$.

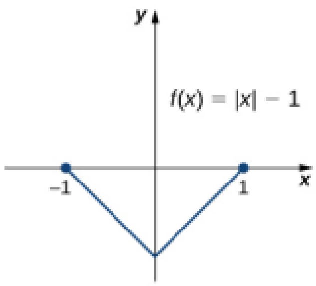

Figure 28 : *No c such tha* $f'(c)$=0

Example 3.3.29 *For* $f(x) = x^3 + 1$, *it is continuous and differentiable on* $[-1, 1]$.

We have $f'(x) = 3x^2$. *And thus* $\exists c \in]-1, 1[: f'(c) = 0$.

However, this does not imply that $f(a) = f(b)$. $f(-1) = 0 \neq f(1) = 2$.

Theorem 3.3.30 (Finite Increment Theorem or Mean Value Theorem) *Let* f *be continuous over the closed interval* $[a, b]$ *and differentiable over the open interval* $]a, b[$. *Then, there exists at least one point* $c \in]a, b[$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Set

$$g(x) := f(x) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a).$$

Then g is continuous on $[a, b]$ and differentiable on $]a, b[$, and

$$g'(x) := f'(x) - \left[\frac{f(b) - f(a)}{b - a}\right].$$

Moreover,$g(a) = f(a)$, and

$$g(b) = f(b) - \left[\frac{f(b) - f(a)}{b - a}\right](b - a) = f(a).$$

Therefore, by Rolle's theorem,

$$\left(\exists c \in]a, b[\, , g'(c) = 0\right) \Leftrightarrow \left(\exists c \in]a, b[\, , f'(c) = \frac{f(b) - f(a)}{b - a}\right).$$

■

Corollary 3.3.31 *If* f *is defined on an interval and* $f'(x) = 0$ *for all* x *in the interval, then* f *is constant there.*

Proof. Let a and b be any two points in the interval with $a \neq b$. Then, by the Mean Value Theorem, there is a point x in $]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) = 0$ for all x in the interval, so

$$0 = \frac{f(b) - f(a)}{b - a},$$

and consequently, f(b) $= f(a)$. Thus the value of f at any two points is the same and f is constant on the interval. ■

Corollary 3.3.32 *If f and g are defined on the same interval and $f'(x) = g'(x)$, then $f = g + c$ for some number $c \in \mathbb{R}$.*

The proof is left as an exercise.

Corollary 3.3.33 *If $f'(x) > 0$ (resp. $f'(x) < 0$) for all x in some interval, then f is increasing (resp. decreasing) on this interval.*

Proof. Consider the case $f'(x) > 0$. Let a and b be any two points in the interval, with $a < b$. By the Mean Value Theorem, there is a point x in $]a, b[$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$

But $f'(x) > 0$ for all x in the interval, so that

$$\frac{f(b) - f(a)}{b - a} > 0.$$

Since $b - a > 0$, it follows that $f(b) > f(a)$, which proves that f is increasing on the interval. ∎

The case $f'(x) < 0$ is left as an exercise.

Example 3.3.34 *Using the Finite Increments Theorem on the function $f(x) = \sin x$, we prove that*

$$\forall x > 0 : |\sin x| \leq |x|.$$

Solution 3.3.35 *The function f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}$, so it is continuous on $[0, x]$ and differentiable on $]0, x[$, according to the Finite Increments Theorem:*

$$(\exists c \in]0, x[) : (f(x) - f(0)) = (x - 0) f'(c)$$

So:

$$\begin{aligned} \sin x = x \cos c &\Rightarrow |\sin x| = |x| |\cos c| \\ &\Rightarrow |\sin x| \leq |x| \quad (|\cos x| \leq 1, \forall x \in \mathbb{R}) \end{aligned}$$

Hence:

$$\forall x > 0 : |\sin x| \leq |x|.$$

Example 3.3.36 *Prove that* $\forall x > 0 : \dfrac{x}{x+1} < \ln(1 + x) < x.$

Solution 3.3.37 *We set :*$f(t) = \ln(1+t) \Rightarrow f'(t) = \dfrac{1}{t+1}$ *is continuous and differentiable on*$]-1,+\infty[$. *Thus,* f *is continuous on* $[0,x]$ *and differentiable on* $]0,x[$. *According to the Finite Increments Theorem:*

$$(\exists c \in]0,x[) : (f(x) - f(0)) = (x-0)f'(c)$$

So,

$$\ln(1+x) = x.\frac{1}{c+1}$$

And we have :

$$0 < c < x \Rightarrow 1 < 1+c < 1+x$$

Which implies:

$$\text{for } x > 0, \frac{x}{1+x} < \frac{x}{1+c} < x,$$

and $\ln(1+x) = \dfrac{x}{1+c}$.
Therefore,

$$\text{for } x > 0, \frac{x}{1+x} < \ln(1+x) < x$$

The next theorem is a generalization of the mean value theorem. It is of interest because of its use in applications.

Theorem 3.3.38 (Cauchy Mean Value Theorem) *If* f *and* g *are continuous on* $[a,b]$ *and differentiable on* $]a,b[$, *then*

$$\exists c \in]a,b[\,,\ [f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c).$$

If $g(b) \neq g(a)$, *and* $g'(c) \neq 0$, *the above equality can be rewritten as*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Note that if $g(x) = x$, we obtain the Mean Value Theorem.

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by

$$h(x) = [f(b) - f(a)]\, g(x) - [g(b) - g(a)]\, f(x).$$

Then

$$h(a) = f(b)g(a) - f(a)g(b) = h(b),$$

so that h satisfies Rolle's theorem. Therefore,

$$\exists c \in]a,b[\,, h'(c) = 0 \Leftrightarrow [f(b) - f(a)]\, g'(c) - [g(b) - g(a)]\, f'(c) = 0.$$

■

3.3.6 Hôspital's Rule:

Eliminate cases of indeterminacy in the form $(\infty - \infty)\,(0 \times \infty)$

It is used to remove cases of indeterminacy in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Theorem 3.3.39 *Let f and g be differentiable functions near x_0 in domain D:*
Where

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Therefore,

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l \Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = l.$$

(supposing l is a defined limit, it could be ∞).

Proof. By the Cauchy Mean ValueTheorem,

$$\frac{f(a+h) - f(a)}{g(a+h) - g(a)} = \frac{f'(a+th)}{g'(a+th)}$$

for some $0 < t < 1$. Now pass to the limit $h \to 0$ to get the result. ■

Example 3.3.40 *1-* $\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \left(\frac{0}{0}\right) \overset{H}{\to} \lim_{x \to 0} \frac{e^x}{\cos x} = 1 \Rightarrow \lim_{x \to 0} \frac{e^x - 1}{\sin x} = 1.$

2- $\lim_{x \to \infty} \frac{\ln x}{e^x} = \left(\frac{\infty}{\infty}\right) \overset{H}{\to} \lim_{x \to \infty} \frac{\frac{1}{x}}{e^x} = \lim_{x \to \infty} \frac{1}{xe^x} = 0 \Rightarrow \lim_{x \to \infty} \frac{\ln x}{e^x} = 0.$

Remark 3.3.41 *The converse of Hôpital's Rule is not true. It is possible for $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ to exist while $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ does not exist (where either f or g is not differentiable at x_0).*

Example 3.3.42

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{\left(\frac{\sin x}{x}\right)} = \frac{0}{1} = 0$$

$$\begin{aligned} \lim_{x \to x_0} \frac{f'(x)}{g'(x)} &= \lim_{x \to 0} \frac{\left[2x \sin \frac{1}{x} - \frac{\cos \frac{1}{x}}{x^2}\left(x^2\right)\right]}{\cos x} \\ &= \lim_{x \to 0} \frac{\left[2x \sin \frac{1}{x} - \cos \frac{1}{x}\right]}{\cos x} \left(\lim_{x \to 0} \cos \frac{1}{x} \text{ does not exist}\right) \end{aligned}$$

So $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ *doesn't exist*

Eliminate cases of indeterminacy in the form $(\infty - \infty)$ **or** $(0 \times \infty)$

To eliminate the indeterminacy cases $(0 \times \infty)$, we apply Hospital's rule, we write it in the form $\frac{0}{0}$or $\frac{\infty}{\infty}$.

* $(0 \times \infty) = \lim_{x \to x_0} f(x) g(x) = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \frac{0}{\frac{1}{\infty}} = \frac{0}{0} \to H.$

* $(\infty \times 0) = \lim_{x \to x_0} f(x) g(x) = \lim_{x \to x_0} \frac{f(x)}{\frac{1}{g(x)}} = \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty} \to H.$

To remove cases of indeterminacy in the form $\infty - \infty$ we use:

* $(\infty - \infty) = \lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} f(x) \left[1 - \frac{f(x)}{g(x)}\right].$

Applying Hôpital's Rule to $\frac{f(x)}{g(x)}$, which is of the form $\frac{\infty}{\infty}$, we have two cases :

a) $\lim_{x \to x_0} \frac{f(x)}{g(x)} \neq 1$. Hence, $\lim_{x \to x_0} f(x) \left[1 - \frac{f(x)}{g(x)}\right] = \infty.$

b) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$.It becomes the indeterminacy of the form $\infty \times 0$.

$$\lim_{x \to x_0} (f(x) - g(x)) = \lim_{x \to x_0} \frac{1 - \frac{g(x)}{f(x)}}{\frac{1}{f(x)}} = \frac{1-1}{\frac{1}{\infty}} = \frac{0}{0}.$$

Or

$$\lim_{x \to x_0} \frac{\frac{f(x)}{1}}{\frac{1}{1 - \frac{g(x)}{f(x)}}} = \frac{\infty}{\frac{1}{1-1}} \frac{\infty}{\infty} \to H.$$

Example 3.3.43 *a)* $\lim_{x \to +\infty} e^{-x} \ln x = (0 \times \infty)$.

$$\lim_{x \to +\infty} \frac{\ln x}{e^x} \overset{H}{\to} \lim_{x \to +\infty} \frac{\lim_{x \to +\infty} \frac{1}{x}}{e^x} = \lim_{x \to +\infty} \frac{1}{xe^x} = 0.$$

b) $\lim_{x\to 0^+} \ln x + \frac{1}{x} = (-\infty + \infty)$.

$$\lim_{x\to 0^+} \ln x + \frac{1}{x} = \lim_{x\to 0^+} \ln x \left(1 + \frac{\frac{1}{x}}{\ln x}\right).$$

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\ln x} = \frac{+\infty}{-\infty} \overset{H}{\rightarrow} \lim_{x\to 0^+} \frac{-\frac{1}{x^2}}{\frac{1}{x}} = \lim_{x\to 0^+} -\frac{1}{x} = -\infty.$$

Therefore $\lim_{x\to 0^+} \ln x + \frac{1}{x} = \lim_{x\to 0^+} \ln x \left(1 + \frac{\frac{1}{x}}{\ln x}\right) = (-\infty)(-\infty) = +\infty.$

3.4 Elementary functions

We now use power series to strictly define the Exponential, Logarithmic, and Trigonometric functions and describe their properties.

3.4.1 Trigonometric functions

Arcsine Function

$$\begin{array}{llll} f: & \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] & \rightarrow & [-1,1] \\ & x & \rightarrow & f(x) = \sin x \end{array}$$

f is continuous, strictly increasing over $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then f is bijective and therefore f^{-1} exists, is continuous and strictly increasing, and we have $f\left(\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right) = [-1,1]$ and

$$\begin{array}{llll} f^{-1}: & [-1,1] & \rightarrow & \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ & y & \rightarrow & f^{-1}(y) = \arcsin y \end{array}$$

from where we have

$$\left(\begin{array}{c} \arcsin y = x \\ -1 \leq y \leq 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{c} \sin x = y \\ -\frac{\pi}{2} \leq x \leq -\frac{\pi}{2} \end{array}\right).$$

Forthermore, the arcsine function is:

- Differentiable on $]-1,+1[$ and

$$\forall y \in \left]-1,1\right[, (\arcsin y)' = \frac{1}{\sqrt{1-y^2}},$$

in fact

$$y \in]-1,1[: \arcsin y = x \Leftrightarrow y = \sin x \text{ and} \quad (\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x}.$$

But we have

$$\begin{aligned} \cos^2 x + \sin^2 x = 1 &\Leftrightarrow \cos x = \pm\sqrt{1 - \sin^2 x} \\ &\Leftrightarrow \cos x = \sqrt{1 - \sin^2(\arcsin y)} \quad (\cos x > 0, \text{ on } \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[) \\ &\Leftrightarrow \cos x = \sqrt{1 - y^2}. \end{aligned}$$

So

$$(\arcsin y)' = \frac{1}{\sqrt{1 - y^2}}, \forall y \in]-1,1[.$$

See figure 29

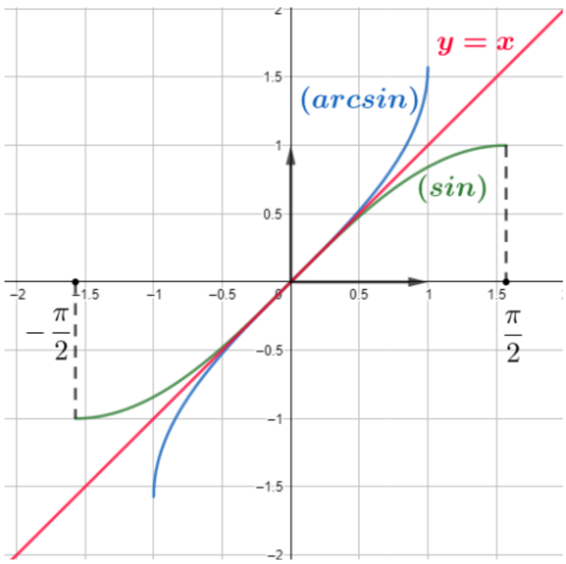

Figure 29 : sin and arcsin

Note

$$\begin{aligned} \sin(\arcsin y) &= y \quad \forall y \in [-1,1]. \\ \arcsin(\sin x) &= x \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned}$$

In other words

$$\sin x = y \Leftrightarrow x = \arcsin y \quad \text{if } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Arccosine Function

$$\begin{aligned} f : [0,\pi] &\to [-1,1] \\ x &\to f(x) = \cos x \end{aligned}$$

f is continuous, strictly decreasing over $[0,\pi]$, then f is bijective and therefore f^{-1} exists, is continuous and strictly decreasing , and we have $f([0,\pi]) = [-1,1]$ and

$$\begin{aligned} f^{-1} : [-1,1] &\to [0,\pi] \\ y &\to f^{-1}(y) = \arccos y \end{aligned}$$

from where we have

$$\begin{pmatrix} \arccos y = x \\ -1 \leq y \leq 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cos x = y \\ 0 \leq x \leq \pi \end{pmatrix}.$$

Forthermore, the arcose function is :

- Differentiable on $]-1,+1[$ and

$$\forall y \in \,]-1,1[\,, \arccos' y = \frac{-1}{\sqrt{1-y^2}}$$

in fact

$$\forall y \in \,]-1,1[\, : \arccos y = x \Leftrightarrow y = \cos x$$

and

$$\begin{aligned} (\arccos y)' &= \frac{1}{(\cos x)'} \\ &= \frac{-1}{\sin x} \quad (\sin x > 0, \text{ on }]0,\pi[) \\ &= \frac{-1}{\sqrt{1-\cos^2 x}} = \frac{-1}{\sqrt{1-y^2}}. \end{aligned}$$

See figure 30

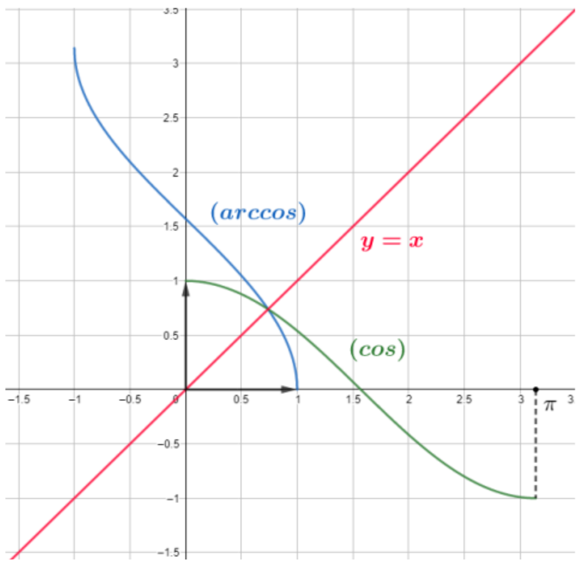

Figure 30 : cos and arccos

Note

$$\begin{aligned} \cos(\arccos y) &= y \ \ \forall y \in [-1,1] \\ \arccos(\cos x) &= x \ \ \forall x \in [0,\pi] \end{aligned}$$

Arctangent function

$$\begin{aligned} f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[& \rightarrow]-\infty,+\infty[\\ x & \rightarrow f(x) = \tan x = \frac{\sin x}{\cos x} \end{aligned}$$

f is continuous, strictly increasing on $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, then f is bijective and therefore f^{-1} exists, is continuous and strictly increasing and we have $f\left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right) =]-\infty,+\infty[$ and

$$\begin{aligned} f^{-1} :]-\infty,+\infty[& \rightarrow \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ y & \rightarrow f^{-1}(y) = \arctan y \end{aligned}$$

from which we have

$$\begin{pmatrix} \arctan y = x \\ y \in \mathbb{R} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \tan x = y \\ -\frac{\pi}{2} < x < \frac{\pi}{2} \end{pmatrix}.$$

Forthermore, the arctangente function is:

- Differentiable on $\mathbb{R}$ and

$$\forall y \in \mathbb{R}, (\arctan)' y = \frac{1}{1+y^2},$$

in fact

$$\forall y \in \mathbb{R} : \arctan y = x \Leftrightarrow y = \tan x$$

and

$$\begin{aligned} (\arctan y)' &= \frac{1}{(\tan x)'} \\ &= \frac{1}{1+\tan^2 x} \\ & \frac{1}{1+y^2}. \end{aligned}$$

See figure

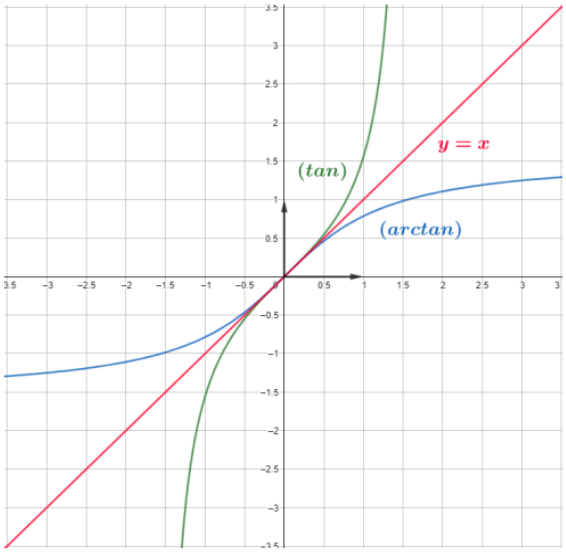

Figure 31 : tan and arctan

Example 3.4.1 *1)*

$$\begin{aligned} \arctan 0 &= \alpha : \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ &\Rightarrow \tan(\arctan 0) = \tan \alpha \\ &\Rightarrow 0 = \tan \alpha : \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\\ &\Rightarrow \alpha = 0. \end{aligned}$$

2)

$$\lim_{y \to -\infty} \arctan y = -\frac{\pi}{2} \ \textit{and} \ \lim_{y \to \infty} \arctan y = \frac{\pi}{2}.$$

Arccotangent function

$$\begin{aligned} f :]0, \pi[&\to]-\infty, +\infty[\\ x &\to f(x) = \cot x = \frac{\cos x}{\sin x} \end{aligned}$$

f is continuous, strictly decreasing on $]0, \pi[$, then f is bijective and therefore f^{-1} exists, is continuous and strictly decreasing and we have $f(]0, \pi[) =]-\infty, +\infty[$ and

$$\begin{aligned} f^{-1} :]-\infty, +\infty[&\to]0, \pi[\\ y &\to f^{-1}(y) = \operatorname{arccot} y \end{aligned}$$

from which we have

$$\begin{pmatrix} \operatorname{arccot} y = x \\ y \in \mathbb{R} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \cot x = y \\ 0 < x < \pi \end{pmatrix}.$$

Forthermore, the arccotangent function is:

- Differentiable on $\mathbb{R}$ and

$$\forall y \in \mathbb{R}, (\operatorname{arccot})' \, y = \frac{1}{1+y^2},$$

in fact

$$\forall y \in \mathbb{R} : \operatorname{arccot} y = x \Leftrightarrow y = \cot x$$

and

$$\begin{aligned} (\operatorname{arccot} y)' &= \frac{1}{(\cot x)'} \\ &= \frac{1}{-1-\cot^2 x} \\ \frac{-1}{1+y^2}. \end{aligned}$$

- Class C^{∞} on $\mathbb{R}$.

See figure 32

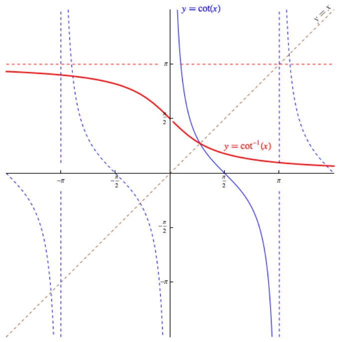

Figure 32 : cot and arccot

1- Show that: $2 \arctan x = \arccos \dfrac{1-x^2}{1+x^2}$

2- Deduce a simplified expression of $\cos(4 \arctan x)$.

3- Solve the equation

$$\arctan x + \arctan 4x = \frac{\pi}{4} - \arctan \frac{1}{5}$$

1- let's as

$$\alpha = \arctan x \Leftrightarrow x = \tan x, \; \alpha \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

and determine

$$\cos(2 \arctan x) = \cos 2\alpha = 2\cos^2 \alpha - 1,$$

hence

$$\cos^2\alpha = \frac{1}{1+\tan^2 x} = \frac{1}{1+x^2},$$

where from

$$\cos 2\alpha = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$

and

$$2\alpha = \arccos\frac{1-x^2}{1+x^2} = 2\arctan x.$$

2- Relationship

$$\begin{aligned}\cos 4\alpha &= 2\cos^2 2\alpha - 1\\ &= 2\cos^2\left(\arccos\frac{1-x^2}{1+x^2}\right) - 1\\ &= 2\left(\frac{1-x^2}{1+x^2}\right)^2 - 1\\ &= \frac{2\left(1-x^2\right)^2 - \left(1+x^2\right)^2}{\left(1+x^2\right)^2}\\ &= \frac{x^4-6x^2+1}{\left(1+x^2\right)^2}.\end{aligned}$$

$\cos\left(4\arctan x\right) = \dfrac{x^4-6x^2+1}{\left(1+x^2\right)^2}.$

$\arctan x + \arctan 4x = \dfrac{\pi}{4} - \arctan\dfrac{1}{5}.$

3- Like

$$\left(\frac{\pi}{4} - \arctan\frac{1}{5}\right) \in \left[0;\frac{\pi}{4}\right],$$

the values of x sought will be such that

$$0 \le \arctan x + \arctan 4x \le \frac{\pi}{4}.$$

So by a formal calculation, let's take the tangent of the two sides

$$\begin{aligned}\tan\left(\arctan x + \arctan 4x\right) & \frac{\tan\left(\arctan x\right) + \tan\left(\arctan 4x\right)}{1 - \tan\left(\arctan x\right)\tan\left(\arctan 4x\right)}\\ &= \frac{x+4x}{1-4x^2} = \frac{5x}{1-4x^2},\end{aligned}$$

$$\tan\left(\frac{\pi}{4}-\arctan\frac{1}{5}\right)=\frac{\tan\left(\frac{\pi}{4}\right)-\tan\left(\arctan\frac{1}{5}\right)}{1+\tan\left(\frac{\pi}{4}\right)\tan\left(\arctan\frac{1}{5}\right)}=\frac{1-\frac{1}{5}}{1+\frac{1}{5}}=\frac{2}{3}.$$

Solution 3.4.2 *Where from*

$$\frac{5x}{1-4x^2}=\frac{2}{3}\Leftrightarrow 2-8x^2=15x\Leftrightarrow 8x^2+15x-2=0.$$

The equation which admits for roots: $\Delta=(15)^2-4\times(-2)\times 8=(17)^2$.

$$x=\frac{-15\pm 17}{16}=\begin{cases} x_1=\frac{1}{8} \\ \text{and} \\ x_2=-2,\ \text{rejected.}\end{cases}$$

Only the solution $x_1=\frac{1}{8}$ *checks for double inequality* $0\le\arctan x+\arctan 4x\le\frac{\pi}{4}$.

3.4.2 Exponential function

Definition 3.4.3 *The exponential function denoted* exp *is the only differentiable function on* $\mathbb{R}$, *equal to its derivative and verifying:* $\exp(0)=1$.

Properties

1. $\forall x\in\mathbb{R}:\exp(x)>0$.
2. $\forall x,y\in\mathbb{R}:\exp(x+y)=\exp(x)\exp(y)$.
3. Euler's notation: We set $\exp(x)=e^x$; where $e^1=e\simeq 2.718$, whence $\forall x,y\in\mathbb{R}:e^{x+y}=e^xe^y$, $e^{-x}=\frac{1}{e^x}$, $e^{x-y}=\frac{e^x}{e^y}$, $(e^x)^n=e^{nx}$.
4. The exp function is strictly increasing on $\mathbb{R}$.
5. $\forall x,y\in\mathbb{R}:\begin{cases} e^x=e^y\Leftrightarrow x=y. \\ e^x<e^y\Leftrightarrow x<y.\end{cases}$
6. The function $x\to e^x$ is a bijection of $\mathbb{R}$ in $\mathbb{R}_+^*$.

Some reference limits:

1. $\lim\limits_{x\to-\infty} e^x = 0,\ \lim\limits_{x\to\infty} e^x = +\infty,\ \lim\limits_{x\to 0}\dfrac{e^x-1}{x} = 1,$

2. $\lim\limits_{x\to 0}\dfrac{e^x}{x^n} = +\infty,\ \lim\limits_{x\to-\infty} x^n e^x = 0$, for all $n \in \mathbb{N}$.

3.4.3 Logarithm function

We call the natural logarithm function denoted ln, the reciprocal function of the exponential function, defined from $]0,+\infty[$ on $\mathbb{R}$ such as

$$\forall x > 0 : x = e^y \Leftrightarrow y = \ln x.$$

Note: The graphs of the natural logarithm function and the exponential function are symmetric with respect to the first bisector, i.e. the line of equation $y = x$, see figure

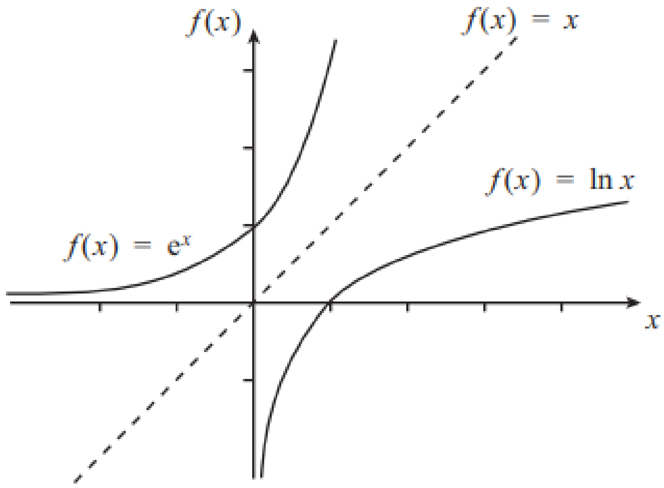

Figure 33 : e^x and $\ln x$

Properties

1. $\ln 1 = 0,\ \ln e = 1.$

2. $\forall x \in \mathbb{R} : \ln e^x = x$ and $\forall x \in]0,+\infty[: e^{\ln x} = x.$

3. The function ln is strictly increasing on $]0,+\infty[$.

4. $\forall x, y \in]0,+\infty[: \ln x = \ln y \Leftrightarrow x = y.$

5. $\forall x, y \in]0,+\infty[: \ln(xy) = \ln x + \ln y.$

6. $\forall x, y \in]0,+\infty[: \ln \dfrac{1}{x} = -\ln x; \ln \dfrac{y}{x} = \ln y - \ln x.$

7. $\forall x \in]0, +\infty[, \forall n \in \mathbb{N} : \ln x^n = n \ln x.$

Some reference limits:

1. $\lim\limits_{x \to 0^+} \ln x = -\infty$, $\lim\limits_{x \to +\infty} \ln x = +\infty$, $\lim\limits_{x \to 0} \dfrac{\ln (x+1)}{x} = 1$,

2. $\lim\limits_{x \to +\infty} \dfrac{\ln x}{x^n} = 0$, $\lim\limits_{x \to -\infty} x^n \ln x = 0$, for all $n \in \mathbb{N}$.

3.4.4 Logarithm function of any base

Definition 3.4.4 *Let a be a strictly positive real number different from 1, we call a logarithm function with base a; the real function denoted* $\log_a$ *and defined on* $]0, +\infty[$ *by*

$$f(x) = \log_a(x) = \frac{\ln(x)}{\ln a},$$

where ln *is the natural logarithm.*

For $a = e$, we find the special case of the natural logarithm function ln, *because* $\ln e = 1$.

If $a = 10$, *then the base* 10 *logarithm function is called the decimal logarithm function, denoted* log *where* $\ln 10 \simeq 2,302$, *it is used in chemistry.*

We also have another logarithm often used in computer science, it is the logarithm in base 2 *where* $\log_2 x = \dfrac{\ln x}{\ln 2}$.

Properties Let a and b be two strictly positive real numbers different from 1, we have:

1. $\log_a 1 = 0$, $\log_a a = 1$, $\log_{\frac{1}{a}} = -\log_a$.

2. $\log_a x = \dfrac{\ln b}{\ln a} \log_b x$, $\forall x > 0$.
   In particular for $a = e$ and $b = 10$, we have $\ln x = \ln 10 \log x$.

3. $\forall x, y \in]0, +\infty[: \log_a x = \log_a y \Leftrightarrow x = y$.

4. $\forall x, y \in]0, +\infty[: \log_a (xy) = \log_a x + \log_a y$.

5. $\forall x, y \in]0, +\infty[: \log_a \left(\dfrac{1}{y}\right) = -\log_a y$, $\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$.

6. $\forall x \in]0, +\infty[, \forall n \in \mathbb{N} : \log_a (x^n) = n \log_a x$.

7. The $\log_a$ function is strictly increasing on $]0, +\infty[$ for $a > 1$ and strictly decreasing on $]0, +\infty[$ for $0 < a < 1$.

3.4.5 Power function

Definition 3.4.5 *Let a be a strictly positive real and different from 1 and x any real, the function a to the power of x or the basic exponential function a is the function denoted a^x and defined by*

$$a^x = e^{x \ln a}.$$

It is the reciprocal function of the $\log_a$ *function (base logarithm a).*

Properties Let a and b be two strictly positive real numbers, and let x and y be two arbitrary real numbers.

1. $a^x > 0$, $\ln a^x = x \ln a$.
2. $1^x = 1$, $a^{x+y} = a^x a^y$, $a^{-x} = \dfrac{1}{a^x}$, $a^{y-x} = \dfrac{a^y}{a^x}$.
3. $(ab)^x = a^x b^x$, $(a^x)^y = a^{xy}$.
4. The base exponential function a is strictly increasing on $\mathbb{R}$ for $a > 1$ and strictly decreasing on $\mathbb{R}$ for $0 < a < 1$.

3.4.6 Hyperbolic functions and their inverses

Hyperbolic sine and cosine

Definition 3.4.6 *The functions hyperbolic sine denoted* sinh *or sh and hyperbolic cosine denoted* cosh *or ch are defined on $\mathbb{R}$ by*

$$\begin{aligned} ch &: \mathbb{R} \to [1, +\infty[, \quad & sh &: \mathbb{R} \to \mathbb{R} \\ x &\to \frac{e^x + e^{-x}}{2} & x &\to \frac{e^x - e^{-x}}{2} \end{aligned}$$

Remark 3.4.7 *Any function $f : I \subset \mathbb{R} - \to \mathbb{R}$ decomposes uniquely into the sum of an even function and of an odd function*

$$\forall x \in I, f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Indeed, $\dfrac{f(x) + f(-x)}{2}$ is even and $\dfrac{f(x) - f(-x)}{2}$ is odd. The hyperbolic cosine and hyperbolic sine functions are respectively the even part and the odd part of the exponential function in this decomposition.

Proposition 3.4.8 *The functions ch and sh are differentiable on $\mathbb{R}$, for all $x \in \mathbb{R}$*

$$ch'(x) = sh(x), \ \ sh'(x) = ch(x).$$

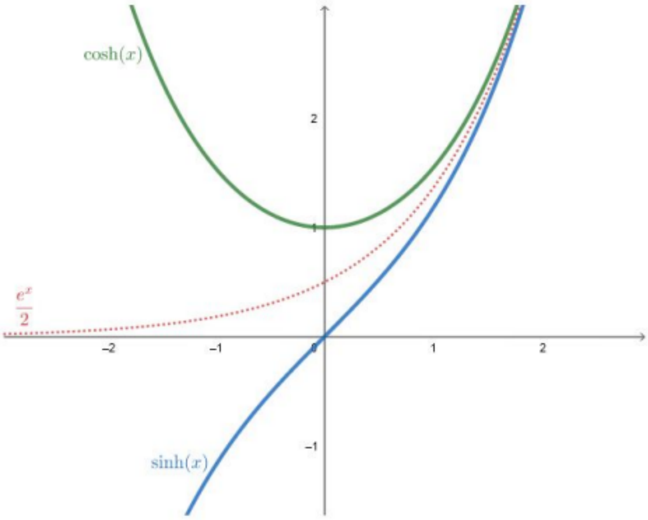

Figure 34 : cosh *and* sinh

Hyperbolic tangent

Definition 3.4.9 *The function hyperbolic tangent denoted* tanh *or th is defined on $\mathbb{R}$ by*

$$\begin{aligned} th : \mathbb{R} &\to \mathbb{R} \\ x &\to thx = \frac{shx}{chx} \end{aligned}$$

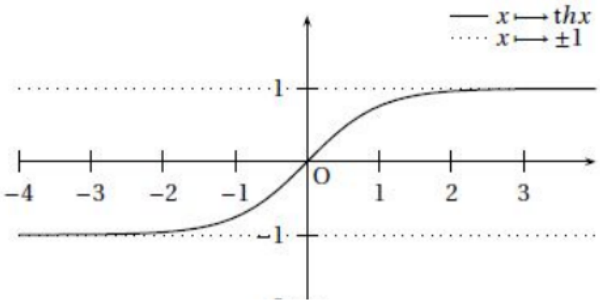

Figure 35 : tanh

Proposition 3.4.10 *The function th is odd, differentiable on $\mathbb{R}$, and for all $x \in \mathbb{R}$*

$$th'x = 1 - th^2x = \frac{1}{ch^2x}.$$

Consequently, th is strictly increasing on $\mathbb{R}$.

Hyperbolic cotangent

Definition 3.4.11 *The function hyperbolic cotangent denoted* coth *or coth is defined on* $\mathbb{R}^*$ *by*

$$\begin{aligned} \cot h : \mathbb{R}^* &\to]-\infty, -1[\cup]1, +\infty[\\ x &\to \cot hx = \frac{chx}{shx} = \frac{1}{thx} = \frac{e^x + e^{-x}}{e^x - e^{-x}} \end{aligned}$$

Proposition 3.4.12 *The function* coth *is odd, differentiable on* $\mathbb{R}^*$*, and for all* $x \in \mathbb{R}^*$

$$\coth' x = 1 - \coth^2 x = \frac{-1}{sh^2 x}.$$

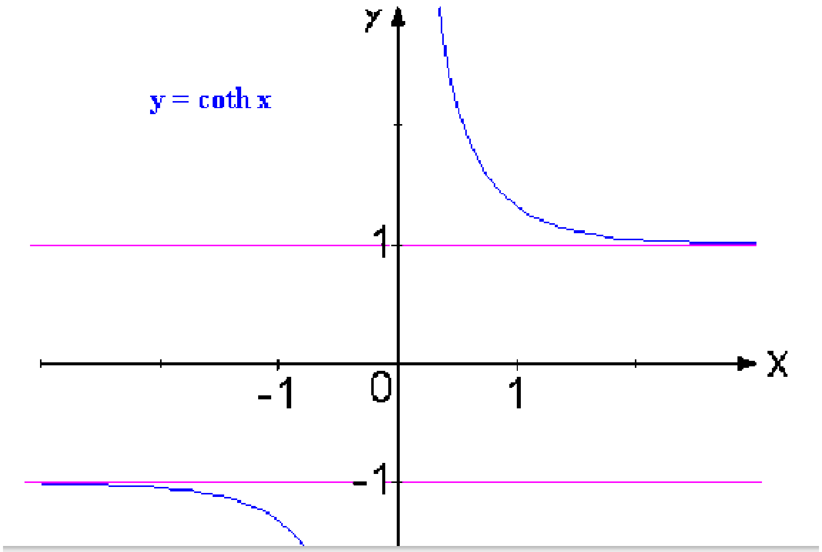

Figure 36 : coth

Proposition 3.4.13 *For all* $x \in \mathbb{R}$

1. $chx + shx = e^x$
2. $chx - shx = e^{-x}$
3. $ch^2x - sh^2x = 1$
4. $ch(x+y) = chxchy + shxshy$
5. $ch(x-y) = chxchy - shxshy$
6. $sh(x+y) = shxchy + chxshy$
7. $sh(x-y) = shxchy - chxshy$
8. $th(x+y) = \dfrac{th(x) + th(y)}{1 + th(x)th(y)}$
9. $th(x-y) = \dfrac{th(x) - th(y)}{1 - th(x)th(y)}$

Hyperbolic sine argument function

Proposition 3.4.14 *The application (*sinh*) $sh : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing so admits a reciprocal function denoted* arg sinh *or* $\arg sh : \mathbb{R} \to \mathbb{R}$,

hence we have
$$\begin{pmatrix} \arg shy = x \\ y \in \mathbb{R} \end{pmatrix} \Leftrightarrow \begin{pmatrix} shx = y \\ x \in \mathbb{R} \end{pmatrix}.$$
Forthermore, the arg sinh function is:

- Differentiable on $\mathbb{R}$ and
$$(\arg sh)' \, y = \frac{1}{\sqrt{y^2+1}},$$

in fact
$$\arg shy = x \Leftrightarrow y = shx$$

and
$$\begin{aligned}(\arg sh)' \, y &= \frac{1}{(shx)'} \\ &= \frac{1}{ch\,(x)} \\ &= \frac{1}{\sqrt{sh^2\,(x)+1}} \\ &= \frac{1}{\sqrt{y^2+1}}.\end{aligned}$$

See Figure 37

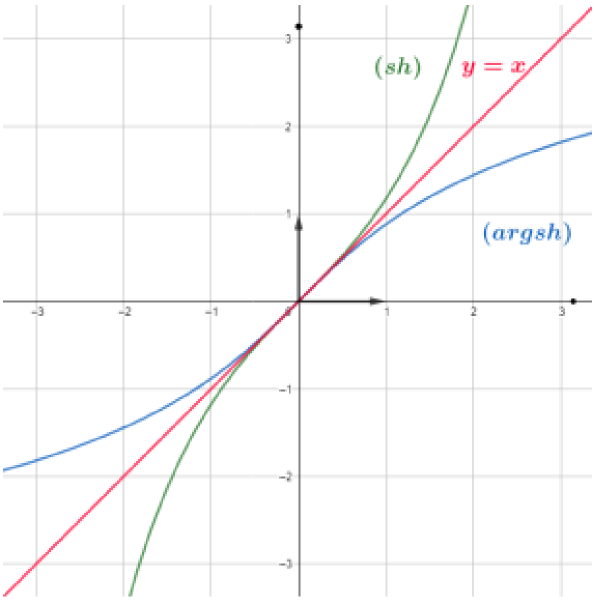

Figure 37 : sh and $\arg sh$

Remark 3.4.15 *Logarithmic expression of* $\arg shy$
$$\forall y \in \mathbb{R} : \arg shy = \ln\left(y + \sqrt{y^2+1}\right).$$

Indeed

$$\forall (y,x) \in \mathbb{R}^2,\ x = \arg shy \Leftrightarrow y = shx \text{ and } chx = +\sqrt{sh^2x+1} = \sqrt{y^2+1}$$

and as

$$\begin{array}{l} e^x = shx + chx = y + \sqrt{y^2+1} \\ x = \ln\left(y + \sqrt{y^2+1}\right). \end{array}$$

Hyperbolic cosine argument function

Proposition 3.4.16 *The application (*cosh*) $ch : [0,+\infty[\to [1,+\infty[$ is continuous and strictly increasing so admits a reciprocal function denoted* $\arg\cosh$ *or* $\arg ch : [1,+\infty[\to [0,+\infty[$,

hence we have

$$\left(\begin{array}{l} \arg chy = x \\ y \geq 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{l} chx = y \\ x \geq 0 \end{array}\right).$$

Forthermore, the $\arg\cosh$ function is:

- Differentiable on $]1,+\infty[$ and

$$(\arg ch)'\, y = \frac{1}{\sqrt{y^2-1}}$$

in fact

$$\arg chy = x \Leftrightarrow y = chx$$

and

$$\begin{aligned} (\arg ch)'\, y &= \frac{1}{(chx)'} \\ &= \frac{1}{sh\,(x)} \\ &= \frac{1}{\sqrt{ch^2\,(x)-1}} \\ &= \frac{1}{\sqrt{y^2-1}}, \end{aligned}$$

for $y \in]1, +\infty[$. See Figure 37

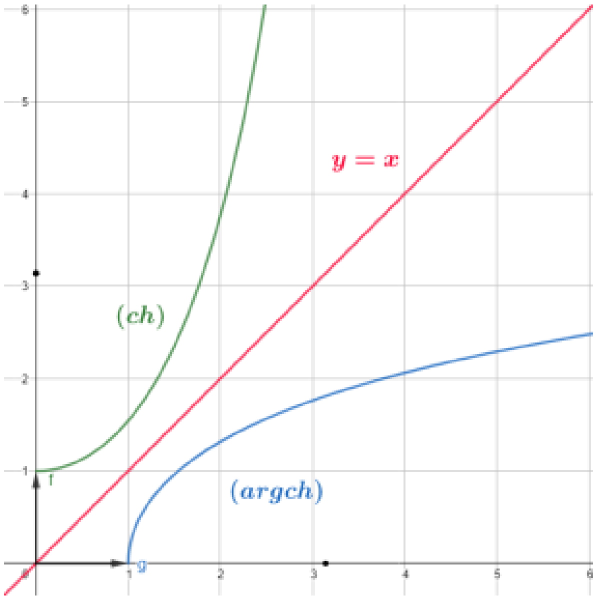

Figure 37 : ch and $\arg ch$

Remark 3.4.17 *Logarithmic expression of* $\arg chy$

$$\forall y \in [1, +\infty[: \arg chy = \ln\left(y + \sqrt{y^2 - 1}\right).$$

Indeed

$$\forall (y, x) \in [1, +\infty[\times[0, +\infty[, \ x = \arg chy \Leftrightarrow y = chx \ \textit{and} \ shx = +\sqrt{ch^2x - 1} = \sqrt{y^2 - 1}$$

and as

$$\begin{aligned} e^x &= chx + shx = y + \sqrt{y^2 - 1} \\ x &= \ln\left(y + \sqrt{y^2 - 1}\right). \end{aligned}$$

Hyperbolic tangent argument function

Proposition 3.4.18 *The application* ($\tanh$) $th : \mathbb{R} \to]-1, 1[$ *is continuous and strictly increasing so admits a reciprocal function noted* $\arg\tanh$ *or* $\arg th :]-1, 1[\to \mathbb{R}$.

hence have

$$\begin{pmatrix} \arg thy = x \\ y \in]1, 1[\end{pmatrix} \Leftrightarrow \begin{pmatrix} thx = y \\ x \in \mathbb{R} \end{pmatrix}.$$

Forthermore, the $\arg\tanh$ function is :

- Differentiable on $]1,1[$ and

$$(\arg th)'\, y = \frac{1}{1-y^2},$$

in fact

$$\arg thy = x \Leftrightarrow y = thx$$

and

$$\begin{aligned}(\arg th)'\, y &= \frac{1}{(thx)'}\\ &= \frac{1}{1-th^2x}\\ &= \frac{1}{1-y^2}.\end{aligned}$$

See Figure 38

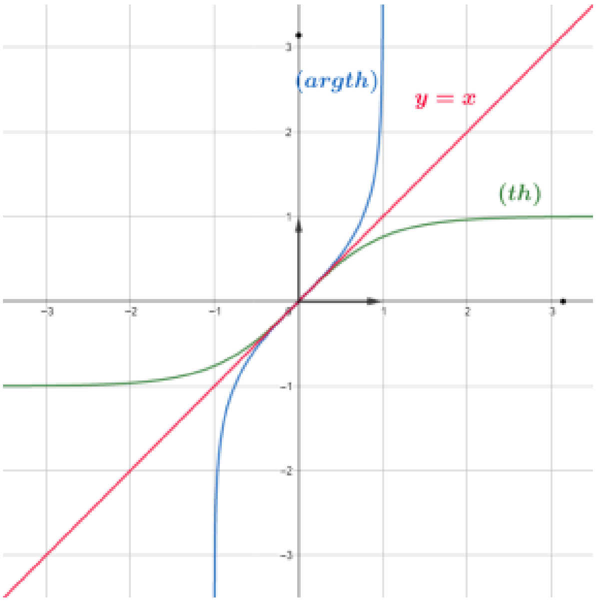

Figure 38 : th and $\arg th$

Remark 3.4.19 *Logarithmic expression of* $\arg thy$

$$\forall y \in \left]-1,1\right[: \arg thy = \frac{1}{2}\ln\frac{1+y}{1-y}.$$

Indeed

$$\begin{aligned}\forall\,(y,x) \in \left]-1,1\right[\times \mathbb{R},\ \ x = \arg thy &\Leftrightarrow y = thx\\ &\Leftrightarrow y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}\\ &\Rightarrow y\left(e^{2x}+1\right) = e^{2x}-1\\ &\Rightarrow e^{2x} = \frac{1+y}{1-y}\\ &\Rightarrow x = \frac{1}{2}\ln\frac{1+y}{1-y}.\end{aligned}$$

Hyperbolic cotangent argument function

Proposition 3.4.20 *The application* $\coth : \mathbb{R}^* \to]-\infty, -1[\cup]1, +\infty[$ *is continuous and strictly decreasing so admits a reciprocal function noted* $\arg\coth$ *or* $\arg cth :]-\infty, -1[\cup]1, +\infty[\to \mathbb{R}^*$,

hence we have

$$\begin{pmatrix} \arg\coth y = x \\ y \in]-\infty, -1[\cup]1, +\infty[\end{pmatrix} \Leftrightarrow \begin{pmatrix} \coth x = y \\ x \in \mathbb{R}^* \end{pmatrix}$$

Forthermore, the $\arg\coth$ function is:

- Differentiable on $]-\infty, -1[\cup]1, +\infty[$ and

$$(\arg\coth)' y = \frac{1}{1 - y^2},$$

in fact

$$\forall y \in]-\infty, -1[\cup]1, +\infty[: \arg\coth y = x \Leftrightarrow y = \coth x$$

and

$$\begin{aligned} (\arg\coth)' y &= \frac{1}{(\coth x)'} \\ &= \frac{1}{1 - \coth^2 x} \\ &= \frac{1}{1 - y^2}. \end{aligned}$$

See Figure 31

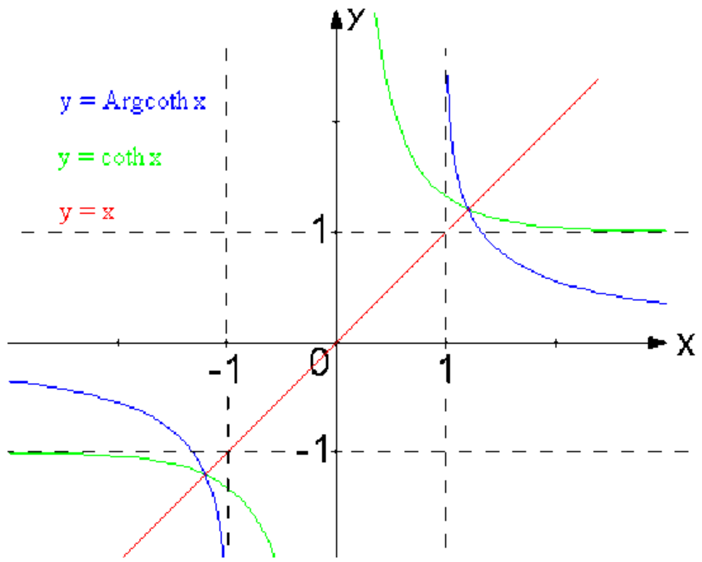

Figure 39 : coth and arg coth

Remark 3.4.21 *Logarithmic expression of* $\arg\coth y$

$$\forall y \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[: \arg\coth y = \frac{1}{2} \ln \frac{y+1}{y-1}.$$

Indeed

$$\begin{aligned} \forall \left(y, x\right) \in \left]-\infty, -1\right[\cup \left]1, +\infty\right[\times \mathbb{R}^*,\ x = \arg\coth y \quad & \Leftrightarrow y = \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} \\ & \Leftrightarrow y = \frac{e^{2x}+1}{e^{2x}-1} \\ & \Leftrightarrow y\left(e^{2x}-1\right) = e^{2x}+1 \\ & \Leftrightarrow e^{2x} = \frac{y+1}{y-1} \\ & \Rightarrow x = \frac{1}{2} \ln \frac{y+1}{y-1}. \end{aligned}$$