**U.F.A.S1. University**
**Faculty of Sciences**

**Department of Mathematics**
**L1 S.M**

# Sheet 2-Maths 1-solutions [1]

## Exercises

Go to : **Solutions**

**Exercise 1** :

Determine the definition domain for each function $f$, $g$, $f \circ g$ and $g \circ f$ with :

1. $f(x) = x^2 - 4, \quad g(x) = \sqrt{x}$
2. $f(x) = \sqrt{x}, \quad g(x) = \dfrac{x-1}{x-2}$

Go to : Solution to Exercise 1

**Exercise 2** :

Determine the lower bound, the upper bound, the infimum, the supremum, the maximum and the minimum for each set if they exist :

$$A = \{x \in \mathbb{R} : x^2 < 2\}, \quad B = \left\{\frac{1}{n};\, n \in \mathbb{N}^*\right\}, \quad C = \left\{\frac{1}{n} + (-1)^n;\, n \in \mathbb{N}^*\right\},$$

$$D = \left\{\frac{2n}{2n-1};\, n \in \mathbb{N}^*\right\}, \quad E = \left\{\frac{x+1}{x+2};\, x \le -3\right\}.$$

Go to : Solution to Exercise 2

**Exercise 3** :

The same questions for the following sets :

$$X = \left\{\frac{2xy}{x^2+y^2};\, x \in \mathbb{R}^*,\, y \in \mathbb{R}^*\right\}, \quad Y = \left\{\frac{(-1)^n}{n} + \frac{2}{n};\, n \in \mathbb{N}^*\right\}.$$

1. Dr. Rachid Chougui, 2 novembre 2025
chougui61@yahoo.fr
Not necessary to solve (*)

Go to : Solution to Exercise 3

**Exercise 4** :

Using the characterization of the least upper bound and the greatest lower bound, show that :

1. $\sup A = \frac{3}{2}$ and $\inf A = 1$ for $A = \left\{ \frac{3n+1}{2n+1} \mid n \in \mathbb{N} \right\}$.
2. $\sup B = 2$ and $\inf B = 0$ for $B = \left\{ \frac{1}{n} + \frac{1}{n^2} \mid n \in \mathbb{N}^* \right\}$.
3. $\sup C = 1$ and $\inf C = 0$ for $C = \{ e^{-n} \mid n \in \mathbb{N} \}$.
4. $\sup D = -1$ and $\inf D = -2$ for $D = \left\{ \frac{1}{n^2} - 2 \mid n \in \mathbb{N}^* \right\}$.

Determine $\max A$, $\min A$, $\max B$, $\min B$, $\max C$, $\min C$, $\max D$ and $\min D$ if they exist.

Go to : Solution to Exercise 4

**Exercise 5** :

1. Show the following inequalities :
   (a) $|x| + |y| \leq |x+y| + |x-y|, \quad \forall x, y \in \mathbb{R}$.
   (b) $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}, \quad \forall x, y \in \mathbb{R}_+$.
   (c) $\left| \sqrt{x} - \sqrt{y} \right| \leq \sqrt{|x-y|}, \quad \forall x, y \in \mathbb{R}_+$.
2. Let $[x]$ denote the integer part of $x$. Show that for all $x, y \in \mathbb{R}$ :
   (a) $x \leq y \Rightarrow [x] \leq [y]$.
   (b) $[x] + [y] \leq [x+y] \leq [x] + [y] + 1$.

Go to : Solution to Exercise 5

**Exercise 6. (It can be solved in Course)** :

1. Prove that :
   (a) The sum of a rational number and an irrational number is irrational.
   (b) $\sqrt{2} \notin \mathbb{Q}$.
   (c) $0.336433643364\ldots \in \mathbb{Q}$.
2. Let $a \in [1, +\infty[$. Simplify

$$x = \sqrt{a + 2\sqrt{a-1}} + \sqrt{a - 2\sqrt{a-1}}.$$

3. Compute :
   (a) $A = \sum_{k=0}^{n} \binom{n}{k}$,
   (b) $B = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right), \quad n \in \mathbb{N}^*$.

Go to : Solution to Exercise 6

## Solutions

Go back to : **Exercises**

**Solution 1** :

Definition domains of the functions $f$, $g$, $f \circ g$, and $g \circ f$ :

1. $D_f = ]-\infty, +\infty[$
2. $D_g = \{x \in \mathbb{R} : x \geq 0\} = [0, +\infty[$
3. Domain of the function $f \circ g$ :

$$x \in D_{f \circ g} \iff x \in D_g \text{ and } g(x) \in D_f.$$

That is :

$$x \in [0, +\infty[ \text{ and } \sqrt{x} \in ]-\infty, +\infty[ \iff x \in [0, +\infty[.$$

Hence,

$$D_{f \circ g} = [0, +\infty[.$$

For all $x \in D_{f \circ g}$,

$$(f \circ g)(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 - 4 = x - 4.$$

4. Domain of the function $g \circ f$ :

$$x \in D_{g \circ f} \iff x \in D_f \text{ and } f(x) \in D_g.$$

That is :

$$x \in ]-\infty, +\infty[ \text{ and } x^2 - 4 \in [0, +\infty[.$$

Hence,

$$x \in ]-\infty, -2] \cup [2, +\infty[,$$

and therefore

$$D_{g \circ f} = ]-\infty, -2] \cup [2, +\infty[.$$

For all $x \in D_{g \circ f}$,

$$(g \circ f)(x) = g(f(x)) = g(x^2 - 4) = \sqrt{x^2 - 4}.$$

Go back to : Exercise 1

**Solution 2** :

**Given sets :**

$$A = \{x \in \mathbb{R} : x^2 < 2\},$$
$$B = \{1/n : n \in \mathbb{N}^*\},$$
$$C = \{1/n + (-1)^n : n \in \mathbb{N}^*\},$$
$$D = \left\{\frac{2n}{2n-1} : n \in \mathbb{N}^*\right\},$$
$$E = \left\{\frac{x+1}{x+2} : x \leq -3\right\}.$$

**Set A.** Since $x^2 < 2$ iff $-\sqrt{2} < x < \sqrt{2}$, we have

$$A = (-\sqrt{2}, \sqrt{2}).$$

Hence $\inf A = -\sqrt{2}$, $\sup A = \sqrt{2}$; neither minimum nor maximum exist (endpoints are not included).

**Set B.** $B = \{1,\ 1/2,\ 1/3, \dots\}$. The sequence decreases to 0. Thus

$$\inf B = 0, \qquad \sup B = 1,$$

with $\max B = 1$ (attained at $n = 1$) and no minimum.

**Set C.** For even $n$, $1/n + (-1)^n = 1 + 1/n$ (decreasing to 1). For odd $n$, $1/n + (-1)^n = -1 + 1/n$ (decreasing to $-1$). The largest value occurs at $n = 2$ and equals $3/2$. The odd subsequence approaches $-1$ from above. Therefore

$$\sup C = \frac{3}{2} \quad \text{(attained at } n = 2\text{)}, \qquad \inf C = -1 \quad \text{(not attained)}.$$

So $\max C = 3/2$ and $\min C$ does not exist.

**Set D.** The terms are $2,\ 4/3,\ 6/5, \dots$ and satisfy $\dfrac{2n}{2n-1} \downarrow 1$. Hence

$$\sup D = 2 \text{ (attained at } n = 1\text{)}, \qquad \inf D = 1 \text{ (limit, not attained)}.$$

Thus $\max D = 2$ and there is no minimum.

**Set E.** Consider $f(x) = \dfrac{x+1}{x+2}$ for $x \le -3$. Since

$$f'(x) = \frac{1}{(x+2)^2} > 0 \quad \text{for } x \neq -2,$$

$f$ is increasing on $(-\infty, -2)$. Therefore on $x \le -3$ the image is

$$\left(\lim_{x \to -\infty} f(x),\ f(-3)\right] = (1, 2].$$

Hence $\inf E = 1$ (not attained), $\sup E = 2$ (attained at $x = -3$), so $\max E = 2$ and no minimum exists.

Go back to : Exercise 2

**Solution 3** :

Let

$$X = \left\{ \frac{2xy}{x^2 + y^2} \ :\ x \in \mathbb{R}^*,\ y \in \mathbb{R}^* \right\}.$$

**(1) Lower bound.** For any nonzero real numbers $x$ and $y$ we have

$$(x+y)^2 \geq 0 \iff x^2+y^2+2xy \geq 0 \iff \frac{2xy}{x^2+y^2} \geq -1.$$

Thus $-1$ is a lower bound of $X$. If we choose $y=-x$ (with $x \neq 0$) then

$$\frac{2x(-x)}{x^2+(-x)^2} = \frac{-2x^2}{2x^2} = -1,$$

so $-1 \in X$. Therefore $\inf X = -1$ and the infimum is attained ; $-1$ is a minimum of $X$.

**(2) Upper bound.** For any nonzero real numbers $x$ and $y$ we have

$$(x-y)^2 \geq 0 \iff x^2+y^2-2xy \geq 0 \iff \frac{2xy}{x^2+y^2} \leq 1.$$

Thus 1 is an upper bound of $X$. If we choose $y=x$ (with $x \neq 0$) then

$$\frac{2xx}{x^2+x^2} = \frac{2x^2}{2x^2} = 1,$$

so $1 \in X$. Therefore $\sup X = 1$ and the supremum is attained ; 1 is a maximum of $X$.

Hence the set $X$ has minimum $-1$ and maximum 1.

Go back to : Exercise 3

**Solution 4** :

1. For $A = \left\{\frac{3n+1}{2n+1},\, n \in \mathbb{N}\right\}$ :
   — $\inf A = 1$ since $3n+1 \geq 2n+1 \Rightarrow \frac{3n+1}{2n+1} \geq 1$ and equality holds for $n=0$. Thus $\min A = 1 = \inf A$.
   — $\sup A = \frac{3}{2}$ because $\frac{3n+1}{2n+1} \leq \frac{3}{2}$ for all $n$, and for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\frac{3}{2} - \varepsilon < \frac{3n_\varepsilon+1}{2n_\varepsilon+1}$. Hence $\sup A = \frac{3}{2}$, but $\frac{3}{2} \notin A$ (no maximum).
2. For $B = \left\{\frac{1}{n} + \frac{1}{n^2},\, n \in \mathbb{N}^*\right\}$ :
   — $\sup B = 2$, since $2 \geq \frac{1}{n} + \frac{1}{n^2}$ for all $n$ and equality holds for $n=1$, so $\max B = 2$.
   — $\inf B = 0$ because all terms are positive and for every $\varepsilon > 0$ we can find $n$ large enough so that $\frac{1}{n} + \frac{1}{n^2} < \varepsilon$. Hence $\inf B = 0$ (no minimum).
3. For $C = \{e^{-n},\, n \in \mathbb{N}\}$ :

— $\sup C = 1$, since $0 \leq e^{-n} \leq 1$ and equality holds for $n = 0$ ($\max C = 1$).
— $\inf C = 0$, as $e^{-n} > 0$ and for any $\varepsilon > 0$ there exists $n$ such that $e^{-n} < \varepsilon$, so $\inf C = 0$ (no minimum).

4. For $D = \left\{ \frac{1}{n^2} - 2,\ n \in \mathbb{N}^* \right\}$ :
— $\sup D = -1$, since $\frac{1}{n^2} - 2 \leq -1$ and equality holds for $n = 1$, so $\max D = -1$.
— $\inf D = -2$, as $\frac{1}{n^2} - 2 > -2$ for all $n$, and for large $n$, $\frac{1}{n^2} - 2$ approaches $-2$ from above. Hence $\inf D = -2$ (no minimum).

Go back to : Exercise 4

**Solution 5** :

1. For all $x, y \in \mathbb{R}$, we have :
—
$$2|x| = |(x+y) + (x-y)| \Rightarrow 2|x| \leq |x+y| + |x-y|$$
and
$$2|y| = |(x+y) + (y-x)| \Rightarrow 2|y| \leq |x+y| + |x-y|.$$
Thus,
$$|x| + |y| \leq |x+y| + |x-y|, \quad \forall x, y \in \mathbb{R}.$$
— For all $x, y \geq 0$, we have :
$$x + y \leq x + 2\sqrt{xy} + y,$$
because $2\sqrt{xy} \geq 0$. Hence
$$x + y \leq (\sqrt{x} + \sqrt{y})^2 \Rightarrow \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}.$$
— For all $x, y \geq 0$, we have
$$x = (x-y) + y \quad \text{and} \quad (x-y) + y \leq |x-y| + y,$$
therefore
$$\sqrt{x} \leq \sqrt{|x-y| + y}.$$
Using (b), we get
$$\sqrt{x} \leq \sqrt{|x-y|} + \sqrt{y} \Rightarrow \sqrt{x} - \sqrt{y} \leq \sqrt{|x-y|}.$$
Similarly,
$$\sqrt{y} \leq \sqrt{|y-x| + x} \Rightarrow \sqrt{y} \leq \sqrt{|x-y|} + \sqrt{x} \Rightarrow \sqrt{x} - \sqrt{y} \geq -\sqrt{|x-y|}.$$
Thus,
$$|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}.$$

2. Let $[x]$ denote the integer part of $x$. Show that for all $x, y \in \mathbb{R}$ :

— If $x \leq y$, then $[x] \leq [x] \leq y < [y] + 1$, hence $[x] \leq [y]$ since $[y]$ is the greatest integer $\leq y$ and $[x]$ is an integer.

— Since
$$[x] \leq x < [x] + 1 \quad \text{and} \quad [y] \leq y < [y] + 1,$$
we get
$$[x] + [y] \leq x + y < [x] + [y] + 2.$$
As $[x + y]$ is the greatest integer $\leq x + y$, we have
$$[x] + [y] \leq [x + y]. \tag{1.5}$$
On the other hand, $[x + y] + 1$ is the smallest integer $> x + y$, so
$$[x + y] + 1 \leq [x] + [y] + 2 \Rightarrow [x + y] \leq [x] + [y] + 1. \tag{1.6}$$
From (1.5) and (1.6) we conclude :
$$[x] + [y] \leq [x + y] \leq [x] + [y] + 1.$$

Go back to : Exercise 5

**Solution 6** :

1.(a) Let $x \in \mathbb{Q}$, $y \notin \mathbb{Q}$. Assume by contradiction that $z = x + y \in \mathbb{Q}$, then $y = z - x \in \mathbb{Q}$, contradiction. Hence, the sum of a rational and an irrational number is irrational.

(b) Suppose, by contradiction, that $\sqrt{2} \in \mathbb{Q}$. Then there exist $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$, such that $\sqrt{2} = \dfrac{p}{q}$, hence
$$p^2 = 2q^2.$$
Thus 2 divides $p^2$, so 2 divides $p$. Let $p = 2k$ for some $k \in \mathbb{Z}$, then
$$4k^2 = 2q^2 \Rightarrow 2k^2 = q^2,$$
so 2 divides $q^2$, hence 2 divides $q$. This contradicts $\gcd(p, q) = 1$. Therefore $\sqrt{2} \notin \mathbb{Q}$.

(c) Let $x = 0.336433643364\ldots$. Then $10^4 x = 3364.33643364\ldots$, so
$$10^4 x - x = 9999x = 3364 \Rightarrow x = \frac{3364}{9999} \in \mathbb{Q}.$$

2. Let $a \in [1, +\infty[$. Simplify
$$x = \sqrt{a + 2\sqrt{a-1}} + \sqrt{a - 2\sqrt{a-1}}.$$
We note that
$$x^2 = 2a + 2\sqrt{a^2 - (a-1)} = 2a + 2\sqrt{a-1}.$$
Hence $x = \sqrt{2(a+1)}$. (This can also be verified algebraically.)

3. Compute :

(a) $A = \sum_{k=0}^{n} \binom{n}{k} = 2^n$.

By the binomial theorem (Binome of Newton),

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus

$$\boxed{A = 2^n.}$$

4. $B = \prod_{k=1}^{n} \left(1 + \frac{1}{k}\right) = n + 1$. Write each factor as a single fraction :

$$1 + \frac{1}{k} = \frac{k+1}{k},$$

so

$$B = \prod_{k=1}^{n} \frac{k+1}{k} = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+1}{n}.$$

All intermediate terms cancel (telescoping product), leaving

$$\boxed{B = n + 1.}$$

Go back to : Exercise 6

**Good Luck !** ■