Maths1 3^{rd} guided works

Exercise 1.

a Using the definition of the limit of a real function, show that:

1. $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}=0$, **2.** $\lim\limits_{x\to 1}\dfrac{1}{(1-x)^2}=+\infty$, **3*.** $\lim\limits_{x\to 0} x\left(x^2+1\right)=0$, **4*.** $\lim\limits_{x\to 1}\dfrac{x+2}{x-1}=+\infty$,
5. $\lim\limits_{x\to +\infty}\dfrac{1}{x^2}=0$, **6*.** $\lim\limits_{x\to 0^+}\dfrac{2}{1+\exp\left\{-\dfrac{1}{x}\right\}}=2$,**7*.** $\lim\limits_{x\to 1}\dfrac{2x^4-6x^3+x^2+3}{x-1}=-8.$

b Calculate the following limits:

1. $\lim\limits_{x\to 1}\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$, **2*.** $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}$, $(n,a)\in\mathbb{N}\times\mathbb{R}$, **3.** $\lim\limits_{x\to 0}\dfrac{\sin mx}{\sin nx}$, $(n,m)\in\mathbb{N}^*\times\mathbb{N}$,
4. $\lim\limits_{x\to +\infty}\left(1+\dfrac{1}{x}\right)^x$, **5*.** $\lim\limits_{x\to +\infty}\left(\dfrac{x-2}{x+2}\right)^x$, **6*.** $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}$, $a\in\mathbb{R}_+^*$, **7.** $\lim\limits_{x\to +\infty}\left(\sin\left(\sqrt{x+1}\right)-\sin\left(\sqrt{x}\right)\right)$,
8. $\lim\limits_{x\to 0} xE(\dfrac{1}{x})$.

c Calculate the following limits using equivalent functions:

1. $\lim\limits_{x\to 0}\dfrac{\tan 2x}{x}$, **2.** $\lim\limits_{x\to 0}\dfrac{2-\cos x-\cos 2x}{\tan^2 x}$, **3*.** $\lim\limits_{x\to 0^+}\ln x.\ln\left[1+\ln 1+x\right].$

Exercise 2:

a Consider a function $f:\mathbb{R}\to\mathbb{R}$ defined by :

$$f(x)=\begin{cases}(ax)^2 & \text{si } x\leq 1,\\ a\sin(\frac{\pi}{2}x) & \text{si } x>1\end{cases}$$

where $a\in\mathbb{R}$ is a real constant. Determine the values of a for which the function f is continuous?

b* Determine all values of the constants α, β, $\gamma\in\mathbb{R}$ such that the following function $g:\mathbb{R}\to\mathbb{R}$

$$g(x)=\begin{cases}1 & \text{if } x\leq 0\\ \alpha e^{-x}+\beta e^{x}+\gamma x\left(e^{x}-e^{-x}\right) & \text{if } 0<x<1\\ e^{2-x} & \text{if } x\geq 1,\end{cases}$$

is continuous:

Exercise 3: Let the function be defined on $\mathbb{R}\setminus\{-1\}$ by:

$$f(x)=\frac{x+1}{x^3+1}$$

Show that f can be extended by continuity at $x=-1$. Specify the value taken by this extension at $x=-1$.

Dr L.Derbal

Exercise 4*: Déterminer les domaines des définitions des fonctions suivantes :

$$\mathbf{1}.x \to \sin\frac{1}{x}, \mathbf{2}.x \to x\sin\frac{1}{x}, \mathbf{3}.x \to \frac{x}{|x|}, \mathbf{4}.\ x \to \frac{x\sin x}{1-\cos x}$$

- Are they continuous on their domains of definition?
- Do they admit continuous extensions?

Exercice 5 Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x\sin x + \cos x$$

1 Show that the equation $f(x) = 0$ has a unique solution in the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

2 Prove that this solution α satisfies $\frac{5\pi}{6} < \alpha < \pi$.

Exercise 6*: Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and let p, q be two strictly positive real numbers. Show that there exists $c \in [a, b]$ such that $pf(a) + qf(b) = (p + q)f(c)$.

Exercise 7: a) Let $f : \mathbb{R} \to \mathbb{R}$, be the function defined by: $f(x) = \frac{\sin x + \cos x}{2 + \cos 2x}$ Show that, for every $a \in \mathbb{R}$, the derivative f' vanishes at least once on the interval $]a, a + 2\pi[$

b) Using the Mean Value Theorem (MVT) on $[x, 2x]$, with $x > 0$, show that:

$$\frac{x}{1+4x^2} < \arctan 2x - \arctan x < \frac{x}{1+x^2}$$

Exercise 8: Using Leibniz's formula, compute the n-th derivative of the function : $x^2 \ln x$ defined on $\mathbb{R}_+^*$.

Exercise 9: a) Solve the equation

$$chx + shx = \frac{1}{shx} - \frac{1}{chx}\ (x \in \mathbb{R}^*)$$

b) Solve the system of equations in $\mathbb{R}$

$$\begin{cases} 3\ln x = 2\ln chy \\ Argchx = 2y \end{cases}$$

Note: Exercises marked with (*) are left for the students.

Dr L.Derbal

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