Model answer key for exercise setnNo.3, Maths1

Solution to exercise 1:

a 1. For every $x \neq 0$, we have :

$$\left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x|$$

Lett $\epsilon > 0$, so there exists $\eta = \epsilon$ such as

$$|x| < \eta \Rightarrow \left|x \sin \frac{1}{x}\right| \leq |x| < \epsilon.$$

2. If $M > 0$, we choose $\eta = \dfrac{1}{\sqrt{M}}$ such as if

$$0 < |x-1| < \eta \Rightarrow \frac{1}{(x-1)^2} > M.$$

3. For all $|x| < 1$, we have :$|x(x^2+1)| \leq |x^3| + |x| \leq |x| + |x| = 2|x|$.
Let $\epsilon > 0$, if we take $\eta = \inf\left\{1, \dfrac{\epsilon}{2}\right\}$, so

$$|x| < \eta \Rightarrow \left|x\left(x^2+1\right)\right| < \epsilon$$

4. Since $x \to 1$we can assume that $|x-1| < 1 \Rightarrow 0 < x < 2$ therefore $|x+2| > 2$,on the other hand if $M > 0$ we choose $\eta = \inf\left\{1, \dfrac{2}{M}\right\}$ such as if $|x-1| < \eta$, we have :

$$|f(x)| > \frac{2}{|x-1|} > M.$$

5. Let $\epsilon > 0$, we find $f(x) = \dfrac{1}{x^2} < \epsilon$ for $x^2 > \dfrac{1}{\epsilon}$ that's to say $|x| > \dfrac{1}{\sqrt{\epsilon}}$, so it is enough to take

$$M = \frac{1}{\sqrt{\epsilon}}.$$

6. Let $\epsilon > 0$, we have :

$$\left|\frac{2}{1+\exp\left\{-\frac{1}{x}\right\}} - 2\right| = \left|\frac{2}{1+\exp\left\{\frac{1}{x}\right\}}\right| < \epsilon \Rightarrow \exp\left\{\frac{1}{x}\right\} > \frac{2}{\epsilon} - 1.$$

If $0 < \epsilon < 2$, we have $0 < x < \dfrac{1}{\ln\left(\frac{2}{\epsilon} - 1\right)}$, so it is enough to take

$$\eta = \frac{1}{\ln\left(\frac{2}{\epsilon} - 1\right)}$$

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If $\epsilon \geq 2$ any value of η will satisfy the requirement (because ... $\dfrac{2}{1+\exp\left\{\dfrac{1}{x}\right\}} < 2$)

7. We have for all $x \neq 1$:

$$\begin{aligned}\frac{2x^4-6x^3+x^2+3}{x-1} &= \frac{(x-1)\left(2x^3-4x^2-3x-3\right)}{x-1} \\ &= 2x^3-4x^2-3x-3\end{aligned}$$

Since $x \to 1$, we can assume that $|x-1|<1$, therefore $0<x<2$, Consequently

$$\begin{aligned}\left|\frac{2x^4-6x^3+x^2+3}{x-1}+8\right| &= \left|2x^3-4x^2-3x+5\right| = |x-1|\left|2x^2-2x-5\right| \\ &\leq |x-1|\left(2\left|x^2\right|+2|x|+5\right) < 17|x-1|\end{aligned}$$

Lett $\epsilon \geq 0$, then there exists $\eta = \inf\left\{1, \dfrac{\epsilon}{17}\right\}$ such that

$$|x-1|<\eta \Rightarrow \left|\frac{2x^4-6x^3+x^2+3}{x-1}+8\right| < \epsilon.$$

b 1. $\lim\limits_{x\to 1}\dfrac{1}{1-x}-\dfrac{2}{1-x^2} = I.F$

1. $\lim\limits_{x\to 0}\dfrac{1}{1-x}-\dfrac{2}{(1-x)(1+x)} = \lim\limits_{x\to 1}\dfrac{(1+x)-2}{(1-x)(1+x)} = \lim\limits_{x\to 1}\dfrac{x-1}{(1-x)(1+x)} = \lim\limits_{x\to 1}\dfrac{1}{1+x} = \dfrac{1}{2}$

2. We have

$$x^n-a^n = (x-a)\left(x^{n-1}+ax^{n-2}+...+a^{n-2}x+a^{n-1}\right)$$

so

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = a^{n-1}+a^{n-1}+...+a^{n-1}+a^{n-1} = na^{n-1}$$

3. We have

$$\lim_{x\to 0}\frac{\sin mx}{\sin nx} = \frac{m}{n}\lim_{x\to 0}\frac{\frac{\sin mx}{mx}}{\frac{\sin nx}{nx}} = \frac{m}{n}$$

4. We have

$$\left(1+\frac{1}{x}\right)^x = \exp\left\{x\ln\left(1+\frac{1}{x}\right)\right\} = \exp\left\{\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}\right\}$$

So

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{y\to 0}\exp\left\{\frac{\ln(1+y)}{y}\right\} = e \qquad \left(\lim_{y\to 0}\frac{\ln(1+y)}{y}=1\right)$$

5. We have

$$\left(\frac{x-2}{x+2}\right)^x = \left(1-\frac{4}{x+2}\right)^x = \frac{\left(\left(1-\frac{1}{y}\right)^y\right)^4}{\left(1-\frac{1}{y}\right)^2},$$

or $\frac{1}{y} = \frac{4}{x+2}$,then

$$\lim_{x\to+\infty}\left(\frac{x-2}{x+2}\right)^x = \lim_{y\to+\infty}\frac{\left(\left(1-\frac{1}{y}\right)^y\right)^4}{\left(1-\frac{1}{y}\right)^2} = e^{-4}$$

6. We have

$$\frac{a^x-1}{x} = \frac{(\exp\{x\ln a\}-1)}{x}$$

Let us set $y = x\ln a$, we find

$$\frac{a^x-1}{x} = \frac{e^y-1}{y}\ln a \Rightarrow \lim_{x\to 0}\left(\frac{a^x-1}{x}\right) = \ln a \lim_{x\to 0}\left(\frac{e^y-1}{y}\right) = \ln a$$

7. We have

$$\sin\left(\sqrt{x+1}\right) - \sin\left(\sqrt{x}\right) = 2\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\cos\frac{\sqrt{x+1}+\sqrt{x}}{2},$$

then

$$0 \le \left|\sin\left(\sqrt{x+1}\right) - \sin\left(\sqrt{x}\right)\right| \le 2\left|\sin\left(\frac{1}{2\left(\sqrt{x+1}+\sqrt{x}\right)}\right)\right| \le \frac{1}{\sqrt{x+1}+\sqrt{x}}$$

(because $\frac{1}{\sqrt{x+1}+\sqrt{x}}$ is verry small). Or

$$\lim_{x\to+\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} = 0$$

we conclude that

$$\lim_{x\to+\infty}\left(\sin\left(\sqrt{x+1}\right) - \sin\left(\sqrt{x}\right)\right) = 0$$

8.

$$\begin{aligned}\frac{1}{x} - 1 < \left[\frac{1}{x}\right] \le \frac{1}{x} \quad &\Rightarrow 1 - x < x\left[\frac{1}{x}\right] \le 1 \\ &\Rightarrow \lim_{x\to 0}(1-x) < \lim_{x\to 0} x\left[\frac{1}{x}\right] \le 1 \\ &\Rightarrow 1 < \lim_{x\to 0} x\left[\frac{1}{x}\right] \le 1 \Rightarrow \lim_{x\to 0} x\left[\frac{1}{x}\right] = 1\end{aligned}$$

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c 1. We know that $\tan 2x \sim 2x$ In the neighborhood of 0, therefore $\lim\limits_{x\to 0}\frac{\tan 2x}{x} = \lim\limits_{x\to 0}\frac{2x}{x} = 2.$

2. We know that $1-\cos x \sim \frac{1}{2}x^2$, $1-\cos 2x \sim \frac{1}{2}(2x)^2$, and $\tan x \sim x$ in the neighborhood of 0, therefore

$$\lim_{x\to 0}\frac{2-\cos x-\cos 2x}{\tan^2 x} = \lim_{x\to 0}\frac{\frac{1}{2}x^2+\frac{1}{2}(2x)^2}{x^2} = \lim_{x\to 0}\frac{\frac{5}{2}x^2}{x^2} = \frac{5}{2}.$$

3. $\ln(1+x) \sim x$, therefore

$$\ln[1+\ln 1+x] \sim \ln(1+x)$$

and we have

$$\ln x.\ln[1+\ln 1+x] \sim \ln x.\ln(1+x) = (x\ln x).\frac{\ln(1+x)}{x}$$
$$\Rightarrow \lim_{x\to 0^+}\ln x.\ln[1+\ln 1+x] = \lim_{x\to 0^+}\left[(x\ln x).\frac{\ln(1+x)}{x}\right] = 0.$$

Corrigé d'exercice 2:

a The function is clearly continuous on $]1,+\infty[$ and on $]-\infty,1[$. For f it is necessary and sufficient that f has a right-hand limit and a left-hand limit at 1, and that these limits are equal. But we have ...

$$\lim_{x\to 1^+} f(x) = a\sin(\frac{\pi}{2}) = a \text{ but } \lim_{x\to 1^-} f(x) = a^2$$

The function f is therefore continuous at 1 if and only if $a^2 = a$, that is, if and only if $a = 1$ or $a = 0$.

b We do the same thing, but this time we need to study the right-hand and left-hand continuity at 0 and at 1, the function g being clearly continuous on $]-\infty,0[$, on$]0,1[$ and on $]1,+\infty[$. On the one hand, we have

$$\lim_{x\to 0^-} g(x) = 1 \text{ and } \lim_{x\to 0^+} g(x) = \alpha+\beta.$$

On the other hand, we have

$$\lim_{x\to 1^-} g(x) = \alpha e^{-1}+\beta e^1+\gamma\left(e^1-e^{-1}\right) \text{ and } \lim_{x\to 1^+} g(x) = e^1.$$

The function g is continuous if and only if the triplet (α,β,γ) satisfies the following system:

$$\begin{cases} \alpha+\beta = 1 \\ \alpha e^{-1}+\beta e^1+\gamma(e^1-e^{-1}) = e^1 \end{cases}$$

We solve this system, for example by removing $e^{-1}L_1$from L_2. We find the equivalent system:

$$\begin{cases} \alpha+\beta = 1 \\ \beta(e^1-e^{-1})+\gamma(e^1-e^{-1}) = e^1-e^{-1}. \end{cases}$$

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We can simplify by $e^1 - e^{-1}$ in the second equation and we find

$$\begin{cases} \alpha + \beta = 1 \\ \beta + \gamma = 1. \end{cases} \Leftrightarrow \begin{cases} \alpha = \gamma \\ \beta = 1 - \gamma \\ \gamma \in \mathbb{R} \end{cases}$$

The set of triplets for which the function g is continuous is therefore given by

$$\{(0,1,0) + \gamma(1,-1,1) : \gamma \in \mathbb{R}\}$$

Solution to exercise 3:
Factor the denominator:

$$x^3 + 1 = (x+1)\left(x^2 - x + 1\right),$$

for $x \neq -1$, we can simplify:

$$f(x) = \frac{x+1}{(x+1)(x^2 - x + 1)} = \frac{1}{x^2 - x + 1}$$

Compute the limit as $x \to -1$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{1}{x^2 - x + 1} = \frac{1}{(-1)^2 - (-1) + 1} = \frac{1}{3}.$$

Since the limit exists and equals $\frac{1}{3}$, f can be extended continuously by defining

$$\tilde{f}(x) = \begin{cases} \dfrac{x+1}{x^3+1}, & x \neq -1 \\ \dfrac{1}{3}, & x = -1 \end{cases}$$

The value taken at $x = -1$ by the continuous extension is $\frac{1}{3}$
Solution to exercise 4:
1. $f(x) = \sin\frac{1}{x}$: The domain of definition is $\mathbb{R}^*$, f is the composition of two continuous functions on $\mathbb{R}^*$,:

$$h : \begin{array}{c} \mathbb{R}^* \to \mathbb{R} \\ x \to y = \dfrac{1}{x} \end{array} \text{ and } g : \begin{array}{c} \mathbb{R} \to \mathbb{R} \\ y \to \sin y \end{array}$$

therefore f is continuous on $\mathbb{R}^*$.
Let's show that f has no limit as $x \to 0$. Let(x_n) and (x'_n), $n \in \mathbb{N}^*$ be the two sequences defined by

$$x_n = \frac{1}{n\pi} \text{ and } x'_n = \frac{1}{2n\pi + \frac{\pi}{2}}$$

We have

$$\lim_{n \to +\infty} x_n = \lim_{n \to +\infty} x'_n = 0.$$

Moreover,

$$f(x_n) = 0 \text{ and } f(x'_n) = 1 \neq 0 \quad \forall n \in \mathbb{N}^* \Rightarrow \lim_{n \to +\infty} f(x_n) = 0 \neq 1 = \lim_{n \to +\infty} f(x'_n)$$

Thus, we have found two sequences (x_n) and (x'_n) that converge to the same limit, such that the limits of $f(x_n)$ and $f(x'_n)$ are different. Therefore, f does not have a limit as $x \to 0$, and it does not admit a continuous extension at 0.

2. $f(x) = x \sin \frac{1}{x}$: The domain of definition is $\mathbb{R}^*$, f is continuous on $\mathbb{R}^*$ (composed of continuous functions). On the other hand,

$$\lim_{x\to 0^+} x \sin \frac{1}{x} = \lim_{x\to 0^-} x \sin \frac{1}{x} = 0$$

Therefore, f admits a continuous extension at $x = 0$. If $\tilde{f}$ is the extended function, we have

$$\tilde{f}(x) = \begin{cases} x \sin \frac{1}{x} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

3. $f(x) = \frac{x}{|x|}$: f is defined and continuous on $\mathbb{R}^*$. Moreover, we have

$$\lim_{x\to 0^+} f(x) = 1 \neq \lim_{x\to 0^-} f(x) = -1$$

Therefore, f does not admit a continuous extension at $x = 0$.

4. $f(x) = \frac{x \sin x}{1 - \cos x}$: f is defined on $\mathbb{R} - \{2k\pi\}$, where $k \in \mathbb{Z}$, and continuous on its domain of definition (as the quotient of two continuous functions with $1 - \cos x \neq 0$). Consequently,

$$\lim_{x\to 2k\pi} \frac{x \sin x}{1 - \cos x} = \lim_{x\to 2k\pi} \frac{2x \sin \frac{x}{2} \cos \frac{x}{2}}{2 \sin \frac{x}{2} \sin \frac{x}{2}} = \lim_{x\to 2k\pi} \frac{x \cos \frac{x}{2}}{\sin \frac{x}{2}}.$$

We distinguish two cases:
Case1: $k = 0$

$$\lim_{x\to 0^+} \frac{x \cos \frac{x}{2}}{\sin \frac{x}{2}} = \lim_{x\to 0^-} \frac{x \cos \frac{x}{2}}{\sin \frac{x}{2}} = 2.$$

Therefore, f admits a continuous extension at $x = 0$.
Case 2: $k \neq 0$, $k \in \mathbb{Z}$

$$\lim_{x\to (2k\pi)^+} \frac{x \cos \frac{x}{2}}{\sin \frac{x}{2}} = \begin{cases} +\infty & \text{if} \quad k > 0 \\ -\infty & \text{if} \quad k < 0 \end{cases}$$

et

$$\lim_{x\to (2k\pi)^-} \frac{x \cos \frac{x}{2}}{\sin \frac{x}{2}} = \begin{cases} -\infty & \text{if} \quad k > 0 \\ +\infty & \text{if} \quad k < 0 \end{cases}$$

Thus, f does not admit a continuous extension at $x = 2k\pi$.

Solution to exercise 5:

1 Existence and uniqueness of a solution on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

a) The function $f(x) = x\sin x + \cos x$ is continuous on all of $\mathbb{R}$ and in particular on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.
Evaluate f at the endpoints

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2} > 0 \\ f\left(\frac{3\pi}{2}\right) &= \frac{3\pi}{2}\sin\left(\frac{3\pi}{2}\right) + \cos\left(\frac{3\pi}{2}\right) = -\frac{3\pi}{2} < 0. \end{aligned}$$

Since f changes sign on the interval, the Intermediate Value Theorem guarantees that there exists at least one solution $\alpha \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$.
b) Prove that the solution is unique. Compute the derivative:

$$f'(x) = x\cos x < 0$$

On the interval $\left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$.Therefore, f is strictly decreasing. Thus, the solution is unique.

2 Prove that $\frac{5\pi}{6} < \alpha < \pi$.

We have $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6}\frac{1}{2} - \frac{\sqrt{3}}{2} = 0.443 > 0$ and $f(\pi) = -1 < 0$.

Since f is strictly decreasing, it crosses zero exactly once, and it must do so between these two values.

Thus $\frac{5\pi}{6} < \alpha < \pi$.

Solution of exercise 6:
Define the function $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - \frac{pf(a) + qf(b)}{p+q}$
We want to show that $g(c) = 0$ for some $c \in [a, b]$. Infact

$$\begin{aligned} g(a) &= f(a) - \frac{pf(a) + qf(b)}{p+q} = \frac{q\left[f(a) - f(b)\right]}{p+q} = \frac{q}{p+q}\left[f(a) - f(b)\right] \\ g(b) &= f(b) - \frac{pf(a) + qf(b)}{p+q} = \frac{p\left[f(b) - f(a)\right]}{p+q} = -\frac{p}{p+q}\left[f(a) - f(b)\right] \end{aligned}$$

So $g(a)$ and $g(b)$ have opposite signs (or at least one is zero).
• If $f(a) = f(b)$, then $g(a) = g(b)$and any $c \in [a, b]$ verified.
• Otherwise, $g(a) < g(b)$ Since f is continuous on $[a, b]$, g is also continuous. By the Intermediate Value Theorem, there exists $c \in [a, b]$ such that

$$\begin{aligned} g(c) = 0 \qquad &\Rightarrow f(c) = \frac{pf(a) + qf(b)}{p+q} \\ &\Rightarrow (p+q) f(c) = pf(a) + qf(b) \end{aligned}$$

Solution to exercise 7: a) The function is 2π-periodic and differentiable on $\mathbb{R}$. For every $a \in \mathbb{R}$, we have $f(a) = f(a + 2\pi)$, and Rolle's theorem shows the existence of a real number $c \in \left]a, a + 2\pi\right[$such as $f'(c) = 0$.

b) Apply MVT to $f(t) = \arctan t$ on $[x, 2x]$:

- $f'(t) = \dfrac{1}{1+t^2}$
- By MVT, there exists $c \in]x, 2x[$ such that:

$$\arctan 2x - \arctan x = f'(c)(2x - x) = \frac{x}{1+c^2}, \text{ for some } c \in]x, 2x[$$

- Since $x < c < 2x$, we have $x^2 + 1 < c^2 + 1 < 4x^2 + 1$
- Taking reciprocals (inequality reverses because the function $t \to \dfrac{1}{t}$ is decreasing for $t > 0$):

$$\frac{1}{1+4x^2} < \frac{1}{1+c^2} < \frac{1}{1+x^2}$$

Multiply through by $x > 0$

$$\frac{x}{1+4x^2} < \frac{x}{1+c^2} < \frac{x}{1+x^2}$$

But $\arctan 2x - \arctan x = \dfrac{x}{1+c^2}$, so finally we get:

$$\frac{x}{1+4x^2} < \arctan 2x - \arctan x < \frac{x}{1+x^2}$$

Solution of Exercise8.
Let us compute, with the aim of applying Leibniz's formula, the successive derivatives of the functions
u and v defined by $u(x) = x^2$ and $v(x) = \ln x$.We have $u'(x) = 2x$, $u''(x) = 2$, then $u^{(k)} = 0$ for all $k \geq 3$. We also have, for all $n \geq 1, v^{(n)}(x) = (-1)^{n-1}(n-1)!x^{-n}$ (this can be shown by induction). According to Leibniz's formula, we have

$$\begin{aligned}
f'(x) &= C_1^0 x^2 \frac{1}{x} + C_1^1 2x \ln x = x + 2x \ln x. \\
f''(x) &= C_2^0 x^2 \left(\frac{-1}{x^2}\right) + C_2^1 2x \frac{1}{x} + C_2^2 2 \ln x = 2 \ln x + 3.
\end{aligned}$$

Then, for $n \geq 3$,

$$\begin{aligned}
f^{(n)}(x) &= C_n^0 x^2 \left((-1)^{n-1}(n-1)!x^{-n}\right) + C_n^1 2x \left((-1)^{n-2}(n-2)!x^{-n+1}\right) + C_n^2 2 \left((-1)^{n-3}(n-3)!x^{-n+2}\right) \\
&= (-1)^{n-1}(n-1)!x^{-n+2} + (-1)^{n-2} 2n(n-2)!x^{-n+2} + (-1)^{n-3} n(n-1)(n-3)!x^{-n+2} \\
&= (-1)^{n-1} x^{-n+2} ((n-1)! - 2n(n-2)! + n(n-1)(n-3)!) \\
&= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} \left(1 - \frac{2n}{n-1} + \frac{n}{n-2}\right) \\
&= \frac{(-1)^{n-1}(n-1)!}{x^{n-2}} \frac{2}{(n-1)(n-2)} \\
&= \frac{2(-1)^{n-1}(n-3)!}{x^{n-2}}.
\end{aligned}$$

Solution of Exercise9. a) $chx + shx = \dfrac{1}{shx} - \dfrac{1}{chx} = \dfrac{chx - shx}{chxshx}$ Dr L.Derbal

Knowing that

$$chx + shx = e^x, chx - shx = e^{-x} \text{ and } chxshx = \frac{1}{2}sh2x = \frac{1}{2}\frac{e^{2x} - e^{-2x}}{2}$$

After substituting into the equation we obtain:

$$\begin{aligned} e^x &= \frac{4e^{-x}}{e^{2x} - e^{-2x}} \Leftrightarrow e^x\left(e^{2x} - e^{-2x}\right) = 4e^{-x} \Leftrightarrow e^{2x}\left(e^{2x} - e^{-2x}\right) = 4 \\ &\Leftrightarrow e^{4x} - 1 = 4 \Leftrightarrow e^{4x} = 5, \text{ hence } x = \frac{\ln 5}{4} \end{aligned}$$

b) The function $Argchx$ being defined on $[1, +\infty[$, the variable x must be ≥ 1. The system is equivalent to:

$$\begin{cases} 3\ln x = 2\ln chy \\ Argchx = 2y \end{cases} \Leftrightarrow \begin{cases} \ln x^3 = \ln ch^2y \\ Argchx = 2y \end{cases} \Leftrightarrow \begin{cases} x^3 = ch^2y \\ x = ch2y \end{cases}$$

Use the identity:

$$ch2y = 2ch^2y - 1 \Leftrightarrow \frac{ch2y + 1}{2} = ch^2y \Leftrightarrow \frac{x+1}{2} = ch^2y,$$

thus

$$\begin{aligned} x^3 &= \frac{x+1}{2} \Leftrightarrow 2x^3 - x - 1 = 0 \Leftrightarrow (x-1)\left(2x^2 + 2x + 1\right) = 0 \\ &\Rightarrow x = 1 \; (2x^2 + 2x + 1 \neq 0 \text{ because the discriminant is negative}) \end{aligned}$$

Hence the only solution of the system: $x = 1$and $y = 0$. Dr L.Derbal