Chapter 5

Vector spaces

In this chapter $\mathbb{K}$ represents a field.

5.1 Vector space

Definition 5.1.1 *Let $\mathbb{K}$ be a commutative field (usually it is $\mathbb{R}$ or $\mathbb{C}$) and let E be a non-empty set with an* ***internal composition law*** *called addition and denoted "+"*

$$\begin{aligned} +: \quad & E \times E \to E \\ & (x, y) \mapsto x + y \end{aligned}$$

and an ***external composition law*** *called multiplication by a scalar and denoted by "·"*

$$\begin{aligned} \cdot: \quad & \mathbb{K} \times E \to E \\ & (\lambda, x) \mapsto \lambda \cdot y \end{aligned}$$

Definition 5.1.2 *A vector space on the field $\mathbb{K}$ or a $\mathbb{K}$- vector space is a triplet $(E, +, .)$ such that:*

1. *(E, +) is a commutative group, where the neutral element is denoted by 0_E and the symmetric of an element x of E will be denoted $-x$.*

2.$\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in E$,

$$\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$$

3. $\forall \alpha, \beta \in \mathbb{K}$, $\forall x \in E$,

$$(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$$

4. $\forall \alpha \in \mathbb{K}$, $\forall x, y \in E$,

$$\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y)$$

5. $1_K \cdot x = x$.

Remark 5.1.3 *1. The elements of E are called vectors and those of $\mathbb{K}$ scalars.*
2. "vector space over $\mathbb{K}$", means $\mathbb{K}$-vector space.

Example 5.1.4 - $(\mathbb{R},+,\cdot)$ *is an* $\mathbb{R}$- *vector space,*
- $(\mathbb{C},+,\cdot)$ *is an* $\mathbb{C}$- *vector space,*
- *If we consider* $\mathbb{R}^n$ *with the following two operations*

$$\begin{array}{lll}(+): & \mathbb{R}^n\times\mathbb{R}^n & \to \mathbb{R}^n\\ & ((x_1,x_2,...,x_n),(y_1,y_2,...,y_n)) & \to (x_1+y_1,x_2+y_2,...,x_n+y_n)\end{array}$$

and

$$\begin{array}{lll}(\cdot): & \mathbb{R}\times\mathbb{R}^n & \to \mathbb{R}^n\\ & (\lambda,(x_1,x_2,...,x_n)) & \to (\lambda x_1,\lambda x_2,...,\lambda x_n)\end{array}$$

we can easily show that $(\mathbb{R}^n,+,.)$ *is an* $\mathbb{R}$- *vector space.*

Example 5.1.5 *The set $E=F(\mathbb{R},\mathbb{R})$ of functions from $\mathbb{R}$ to $\mathbb{R}$ endowed with the usual laws, addition of fuctions and multiplication of the functions by a real number:*

$$(f+g)(x)=f(x)+g(x)$$

and

$$(\alpha.f)(x)=\alpha.f(x),$$

is a $\mathbb{R}$ - *vector space.*

Proposition 5.1.6 *If E is $\mathbb{K}$- vector space, then we have the following properties:*
(1) $\forall x\in E,\ 0_{\mathbb{K}}.x=0_E$,
(2) $\forall x\in E,\ (-1_{\mathbb{K}}).x=-x$
(3) $\forall\lambda\in\mathbb{K},\ \lambda 0_E=0_E$
(4) $\forall\lambda\in\mathbb{K},\ \forall x,y\in E,\ \lambda.(x-y)=\lambda.x-\lambda.y$
(5) $\forall\lambda\in\mathbb{K},\ \forall x\in E,\ \lambda.x=0_E\Leftrightarrow\lambda=0_{\mathbb{K}}$ *or* $x=0_E$.

5.1.1 Vector subspace

In this part, E will denote a $\mathbb{K}$-vector space.

Definition 5.1.7 *A subset F of E is called a vector subspace of E if*
(i) $\emptyset\neq F\subset E$,
(ii) F is a $\mathbb{K}$-vector space with respect to the same laws.

Remark 5.1.8 *1) When* $(F,+,.)$ *is a vector subspace of* E *then* $0_E \in F$.
2) If $0_E \notin F$. *then* $(F,+,.)$ *cannot be a vector subspace of* E.

Theorem 5.1.9 *Let* F *be a nonempty subset of* E*, the following assertions are equivalent :*
(1) F *is a vector subspace of over* $\mathbb{K}$,
(2) F *is stable for addition and for multiplication by a scalar .i.e*

$$\forall \lambda \in \mathbb{K},\ \forall x, y \in F,\ \ \lambda x \in F \text{ and } x + y \in F.$$

(3) $\forall \lambda, \mu \in \mathbb{K},\ \forall x, y \in F,\ \ \lambda x + \mu y \in F.$

Theorem 5.1.10 *A subset* F *of* E *is called a vector subspace of* E *if the following condition hold :*
(i) $0_E \in F$,
(ii) $\forall x, y \in F,\ x + y \in F$,
(ii) $\forall \alpha \in \mathbb{K},\ \forall x \in F,\ \alpha x \in F$.

Example 5.1.11 *(1)* E *and* 0_E *are vector subspaces of* E.
(2) $F = \left\{(x, y) \in \mathbb{R}^2 / x + y = 0\right\}$ *is a vector subspace of* $\mathbb{R}^2$ *over* $\mathbb{R}$ *because ,*
- $0_E = 0_{\mathbb{R}^2} = (0, 0) \in F \Rightarrow F \neq \emptyset$
- $\forall\, (x, y),\ (x', y') \in F,\ \forall \alpha, \beta \in \mathbb{R} : \alpha\,(x, y) + \beta\,(x', y') \in F$, *i.e* $(\alpha x + \beta x', \alpha y + \beta y') \in F$*, we have*

$$(x, y) \in F \Rightarrow x + y = 0 \text{ and } (x', y') \in F \Rightarrow x' + y' = 0$$
$$\alpha x + \beta x' + \alpha y + \beta y' = \alpha\,(x + y) + \beta\,(x' + y') = \alpha\,(0) + \beta\,(0) = 0$$

Then $\alpha\,(x, y) + \beta\,(x', y') \in F$*, so* F *is vector subspace of* $\mathbb{R}^2$.
(3).The set $F = \left\{(x, y) \in \mathbb{R}^2 / x - y + 1 = 0\right\}$ *is not a vector subspace of* $\mathbb{R}^2$ *because the zero vector* $0_{\mathbb{R}^2}$ *does not belong to* F.

5.1.2 Intersection and union of vector subspaces

Proposition 5.1.12 *The intersection of two vector sub-spaces is a vector subspace.*

Proof. Consider F_1 and F_2 two vector subspaces of E. First $0_E \in F_1$, because F_1 is a vector subspace of E. Similarly, $0_E \in F_2$. Thus, $0_E \in F_1 \cap F_2$ and $F_1 \cap F_2$ is therefore not empty. Given x, $y \in F_1 \cap F_2$ and $\alpha, \beta \in \mathbb{R}$, then we have $\alpha x + \beta y \in F_1$ since F_1 is a vector subspace of E. Similarly, $\alpha x + \beta y \in F_2$. Thus, $\alpha x + \beta y \in F_1 \cap F_2$. It follows that $F_1 \cap F_2$ is a vector subspace of E. ■

Lemma 5.1.13 *The intersection $\cap_{i=1}^{n} F_i$ of n vector subspaces of a vector space E ($n \geq 2$, $n \in \mathbb{N}$) is a vector subspace of E.*

Remark 5.1.14 *The union of two vector subspaces is not necessarily a vector subspace.*

Example 5.1.15 *Let $F_1 = \{(x, y) \in \mathbb{R}^2, x = 0\}$ and $F_2 = \{(x, y) \in \mathbb{R}^2, y = 0\}$ two vector subspaces in $\mathbb{R}^2$, $F_1 \cup F_2$ is not a vector subspace, because $u_1 = (0, 1) \in F_1$, $u_2 = (1, 0) \in F_2$ and $u_1 + u_2 = (1, 1) \notin F_1 \cup F_2$.*

5.1.3 Sum of two vector subspaces

Definition 5.1.16 *Let E_1, E_2 be two vector subspaces of a $\mathbb{K}$-vector space E, we call the sum of the two vector subspaces E_1 and E_2 that we denote $E_1 + E_2$ the following set:*

$$E_1 + E_2 = \{x \in E : \exists x_1 \in E_1, \exists x_2 \in E_2 \text{ such that } x = x_1 + x_2\}.$$

Example 5.1.17 *Let $E_1 = \{(x, y) \in \mathbb{R}^2, x = 0\}$ and $E_2 = \{(x, y) \in \mathbb{R}^2, y = 0\}$ vector subspaces in $\mathbb{R}^2$, if $(x, y) \in \mathbb{R}^2$, then*

$$(x, y) = \underset{\in E_1}{(x, 0)} + \underset{E_2}{(0, y)},$$

so $(x, y) \in E_1 + E_2$, hence $E_1 + E_2 = \mathbb{R}^2$.

Proposition 5.1.18 *The sum of two vector subspaces E_1 and E_2 (of the same $\mathbb{K}$ -vector space) is a vector subspace of E containing $E_1 \cup E_2$, i.e., $E_1 \cup E_2 \subset E_1 + E_2$.*

5.1.4 Direct sum of two vector subspaces

Definition 5.1.19 *Let E_1 and E_2 be two vector subspaces of the same $\mathbb{K}$-vector space E. We will say that the sum: $E_1 + E_2$ of two vector subspaces is direct if $E_1 \cap E_2 = \{0_E\}$. We write $E_1 \oplus E_2$.*

Proposition 5.1.20 *Let E_1 and E_2 be two vector subspaces of the same $\mathbb{K}$-vector space E. The sum $E_1 + E_2$ is direct if $\forall x \in E_1 + E_2$, there exists a single vector $x_1 \in E_1$, a single vector $x_2 \in E_2$, such that $x = x_1 + x_2$.*

Example 5.1.21 *Let $F_1 = \{(x, y, z) \in \mathbb{R}^3 : x = 0\}$ and $F_2 = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$ be two vector subspaces in $\mathbb{R}^3$.*

- Let $(x, y, z) \in \mathbb{R}^3$, then $(x, y, z) = \underset{\in F_1}{(0, y, z)} + \underset{\in F_2}{(x, 0, 0)}$, so $(x, y, z) \in F_1 + F_2$, hence $F_1 + F_2 = \mathbb{R}^3$.

- Let $(x, y, z) \in F_1 \cap F_2$, then $(x, y, z) \in F_1$ and $(x, y, z) \in F_2$, this means that $x = 0$ and $y = z = 0$, then $(x, y, z) = 0_{\mathbb{R}^3}$, i.e. $F_1 \cap F_2 = \{0_{\mathbb{R}^3}\}$.

Finally, we conclude that $\mathbb{R}^3 = F_1 \oplus F_2$.

5.1.5 Generating, free, and basis families

Linear combination

Definition 5.1.22 *For $n \in \mathbb{N}^*$, A linear combination of vectors $u_1, u_2, ..., u_n$ of a $\mathbb{K}$-vector space E, is a vector which can be written $V = \sum_{i=1}^{n} \lambda_i u_i$. The elements $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ are called coefficients of the linear combination.*

Example 5.1.23 *In $\mathbb{R}^2$, the vector $U = (9, 8)$ is a linear combination of vectors $(1, 2)$ and $(3, 1)$ because*

$$U = (9, 8) = 3(1, 2) + 2(3, 1)$$

Remark 5.1.24 • *If F is a vector subspace of E, and $u_1, u_2, ..., u_n \in F$, then any linear combination of $u_1, u_2, ..., u_n$ is in F.*

• *Let $u_1, u_2, ..., u_n$, n vectors of a $\mathbb{K}$-vector space E. One can always write 0_E as a linear combination of these vectors, because it suffices to take all zero coefficients of the linear combination.*

• *If $n = 1$, then $V = \lambda_1 u_1$ we say that V is collinear with u_1..*

Generating (Spanning) family

Definition 5.1.25 *We consider a nonempty family $A = (u_1, u_2, ..., u_n)$ of vectors of a $\mathbb{K}$-vector space E with $n \in \mathbb{N}^*$. We say that A generates (spans) E, or that it is generator of E if and only if*

$$Span\{u_1, u_2, ..., u_n\} = E.$$

In other words, any vector of E is a linear combination of the elements of A.

Notation 4 *Given the vectors $u_1, u_2, ..., u_n$ of $\mathbb{K}$-vector space E, we denote $Span(u_1, u_2, ..., u_n)$ or $\langle u_1, u_2, ..., u_n \rangle$ the set of linear combination of $u_1, u_2, ..., u_n$. So we write :*

$$\langle u_1, u_2, ..., u_n \rangle = Span\{u_1, u_2, ..., u_n\} = \left\{ u \in E / \exists \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}; u = \sum_{i=1}^{n} \lambda_i u_i \right\}.$$

Example 5.1.26 *$A = \{u_1 = (1, 0, 0), u_2 = (0, 1, 0), u_3 = (0, 0, 1)\}$ generates $\mathbb{R}^3$, because for all $U = (x, y, z) \in \mathbb{R}^3$ we have:*

$$(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1).$$

Example 5.1.27 *In* $\mathbb{R}^2$*, we consider the vectors* $u_1 = (1,1)$*,* $u_2 = (1,0)$ *and* $u_3 = (0,-1)$*. Let us check that the family* (u_1, u_2, u_3) *generates* $\mathbb{R}^2$*. Let* $X = (x,y) \in \mathbb{R}^2$*, we seek if there exists* $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^2$ *such that* $X = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3$*.*

$$X = (x,y) = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 \quad \Leftrightarrow \begin{cases} x = \lambda_1 + \lambda_2 \\ y = \lambda_1 - \lambda_3 \end{cases}$$
$$\Leftrightarrow \begin{cases} \lambda_2 = x - \lambda_1 \\ \lambda_3 = \lambda_1 - y. \end{cases}$$

We therefore obtain $X = \lambda_1 u_1 + (x - \lambda_1)\, u_2 + (\lambda_1 - y)\, u_3$*, with* $\lambda_1 \in \mathbb{R}$*. So* (u_1, u_2, u_3) *is a generating family of* $\mathbb{R}^2$*.*

Free families

Definition 5.1.28 *We consider a nonempty family* $A = (u_1, u_2, ..., u_n)$ *of* E *with* $n \in \mathbb{N}^*$*. We say that* A *is free if and only if the null vector* 0_E *is a linear combination of elements of* A *unique way. In other words:*

$$\forall \lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}, \ \sum_{i=1}^{n} \lambda_i u_i = 0_E \Rightarrow \lambda_1 = \lambda_2 = ... = \lambda_i = 0_{\mathbb{K}}.$$

Example 5.1.29 *The set* $A = \{u_1 = (1,0,1), u_2 = (0,2,2), u_3 = (3,7,1)\}$ *is free.*

Indeed, let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$*, we have*

$$\lambda_1(1,0,1) + \lambda_2(0,2,2) + \lambda_3(3,7,1) = 0_{\mathbb{R}^3} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0_{\mathbb{R}}.$$

Remark 5.1.30 *We can use the following expressions:*

- *If* A *is free then we also say that the vectors* $u_1, u_2, ..., u_n$ *are linearly independent.*
- *If* A *is not free, we say that* A *is linked.*
- *A family of a single vector is free if and only if this vector is non-zero.*

Example 5.1.31 *In* $\mathbb{R}^2$*, the vector* $u = (2,1)$ *is not collinear with* $v = (1,1)$*, that is to say is free.*

Indeed: let $(\lambda_1, \lambda_2) \in \mathbb{R}^2$*, such that*

$$\lambda_1 u + \lambda_2 v = 0_{\mathbb{R}^2} \Leftrightarrow \begin{cases} 2\lambda_1 + \lambda_2 = 0 \\ \lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = 0_{\mathbb{R}}.$$

The unique solution found is the trivial solution $(0,0)$, the family (u,v) is therefore free.

Example 5.1.32 *In* $\mathbb{R}^2$*, the vectors* $u = (1,2)$*,* $v = (3,4)$ *and* $w = (5,6)$ *are linearly dependent.*

Indeed: let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$*, such that*

$$\lambda_1 u + \lambda_2 v + \lambda_3 w = 0_{\mathbb{R}^2} \Leftrightarrow \begin{cases} \lambda_1 + 3\lambda_2 + 5\lambda_3 = 0 \\ 2\lambda_1 + 4\lambda_2 + 6\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_1 + 3\lambda_2 + 5\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_2 + 2\lambda_3 = 0 \\ \lambda_1 + 2\lambda_2 + 3\lambda_3 = 0 \end{cases}$$
$$\Longleftrightarrow \begin{cases} \lambda_2 = -2\lambda_3 \\ \lambda_1 = \lambda_3. \end{cases}$$

So, this system admits at least one non-trivial solution, for example:

$$\lambda_1 = 1,\ \lambda_2 = -2 \text{ and } \lambda_3 = 1.$$

Since $u - 2v + w = 0_{\mathbb{R}^2}$*, the family* $\{u, v, w\}$*is linearly dependent*

Basis

Definition 5.1.33 *Let* E *be a vector space over a field* $\mathbb{K}$*. A family*

$$A = (u_1, u_2, ..., u_n)$$

is called a basis of E *if it is linearly independent and generating.*

Equivalently, A *is a basis of* E *if and only if every vector* $u \in E$ *can be written in a unique way as a linear combination of the vectors in* A *:*

$$\forall u \in E,\ \exists!\, (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{K}^n, \text{ such that } u = \sum_{i=1}^{n} \lambda_i u_i.$$

The scalars $\lambda_1, \lambda_2, ..., \lambda_n$ *are called the coordinates of* u *in the basis* A*.*

Example 5.1.34
- $B_1 = \{(1,0), (0.1)\}$ *is the canonical basis of* $\mathbb{R}^2$*.*
- $B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$ *is the canonical basis of* $\mathbb{R}^3$*.*
- $B_3 = \{(1,0,...,0), (0,1,0,...,0), ..., (0,...,0,1)\}$ *is the canonical basis of* $\mathbb{R}^n$*.*

Example 5.1.35 *Consider the vector space of real polynomials of degree less than or equal to* 2*.*

$$\mathbb{R}_2[x] = \left\{P(x) = a + bx + cx^2 \ / \ a, b, c \in \mathbb{R}\right\}$$

We claim that the family

$$\mathcal{B} = \left\{P_1(x) = 1,\ P_2(x) = x,\ P_3(x) = x^2\right\}$$

is a basis of $\mathbb{R}_2[x]$.

In fact,

i) *Linear independence*

Let $\alpha,\ \beta,\ \gamma \in \mathbb{R}$, *suppose that*

$$\forall x \in \mathbb{R},\ \alpha P_1(x) + \beta P_2(x) + \gamma P_3(x) = 0.$$

This is equivalent to

$$\forall x \in \mathbb{R},\ \alpha + \beta x + \gamma x^2 = 0.$$

(Since a polynomial that is identically zero must have all coefficients equal to zero, we obtain.

$$\alpha = \beta = \gamma = 0$$

Hence, $\left\{1,\ x,\ x^2\right\}$ *is a linearly independent (free) family.*

ii) Generating property

Let $P \in \mathbb{R}_2[x]$, *by definition, there exist* $a,\ b,\ c \in \mathbb{R}$, *such that*

$$\forall x \in \mathbb{R},\ P(x) = a + bx + cx^2 = aP_1(x) + bP_2(x) + cP_3(x),$$

or equivalently,

$$P = aP_1 + bP_2 + cP_3.$$

Therefore, $\left\{1,\ x,\ x^2\right\}$ *generates* $\mathbb{R}_2[x]$.

Example 5.1.36 *Let*

$$u_1 = (1, 1),\ u_2 = (1, 0),\ u_3 = (0, -1)$$

be vectors in $\mathbb{R}^2$. *As seen in the previous example, the family* (u_1, u_2, u_3) *is a generating family of* $\mathbb{R}^2$ *However, this family is linearly dependent (linked), since*

$$u_1 + u_3 = u_2,$$

which yields a non-trivial linear relation between the vectors. Therefore, (u_1, u_2, u_3) *is not a basis of* $\mathbb{R}^2$. *On the other hand, the family* (u_1, u_2) *is both linearly independent and generating in* $\mathbb{R}^2$. *Consequently,* (u_1, u_2) *is a basis of* $\mathbb{R}^2$.

Example 5.1.37 *Let F be the subset of $\mathbb{R}^3$ defined by:*

$$F = \left\{(x, y, z) \in \mathbb{R}^3 \ / \ x = -2y + z\right\}$$

F is therefore a vector subspace of $\mathbb{R}^3$ generated by the vectors

$$u = (-2, 1, 0) \ \text{and} \ v = (1, 0, 1) .$$

Indeed

$$\begin{aligned} F = \left\{(x, y, z) \in \mathbb{R}^3 \ / \ x = -2y + z\right\} &= \left\{(-2y + z, y, z) \ / \ (y, z) \in \mathbb{R}^2\right\} \\ &= \left\{y(-2, 1, 0) + z(1, 0, 1) \ / \ (y, z) \in \mathbb{R}^2\right\} \\ &= Span\left\{(-2, 1, 0), (1, 0, 1)\right\} = \langle(-2, 1, 0), (1, 0, 1)\rangle \end{aligned}$$

Furthermore, these vectors form a free family so (u, v) is a basis of F.

Proposition 5.1.38 *Let E be a vector space. If*

$$\{e_1, e_2, ..., e_n\} \ \text{and} \ \{u_1, u_2, ..., u_m\}$$

are two bases of E, then $n = m$.

Remark 5.1.39 *If a vector space E admits a basis, then all the bases of E have the same number of elements, this number does not depend on the basis but it only depends on the space E. This common number is called the dimension of E.*

5.1.6 Dimension of vector spaces

Definition 5.1.40 *Let E be a vector space over a field $\mathbb{K}$, and let $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ be a basis of E, The dimension of E, denoted $\dim(E)$, is defined as*

$$\dim(E) = Card(B),$$

that is, the number of elements of the basis $\mathcal{B}$.

Example 5.1.41 *Let*

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

The family $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$, called the canonical (standard) basis.

Therefore,

$$\dim(\mathbb{R}^3) = Card(\{e_1, e_2, e_3\}) = 3.$$

Example 5.1.42 *In the vector space $\mathbb{R}_2[x]$, the family $\left\{1,\ x,\ x^2\right\}$ is a basis. Therefore,*

$$\dim\left(\mathbb{R}_2[x]\right) = Card\left\{1,\ x,\ x^2\right\} = 3.$$

Theorem 5.1.43 *Let E be a vector space of dimension n, then :*

1) Characterization of a basis: A family $\{e_1,\ e_2,\ ...,e_n\}$ of n vectors in E is the basis of E if and only if it is either: generating, or linearly independent (free). That is,

$$\{e_1,\ e_2,\ ...,e_n\} \text{ is abasis} \Leftrightarrow \text{it is generating} \Leftrightarrow \text{it is free.}$$

2) Families with more than n vectors: Let $\{e_1,\ e_2,\ ...,\ e_p\}$ be p vectors in E, with $p > n$, then :

- *The family cannot be free (it is linearly dependent).*
- *If the family is generating, then there exists a subset of n vectors among them that forms a basis of E.*

3) Families with fewer than n vectors: Let $\{e_1, e_2, ..., ep\}$ be p vector in E, with $p < n$, then :

- *The family cannot be generating (it does not span E).*
- *If the family is free, it is possible to find $(n-p)$ additional vectors $\{e_{p+1},\ e_{p+2},\ ...,e_n\}$ in E such that $\{e_1,\ e_2,\ ...,\ e_{p+1},\ ...,e_n\}$ forms a basis for E.*

4) If F is a vector subspace of E : then $\dim F \leq n$, and moreover $\dim F = n \Leftrightarrow F = E$.

Proposition 5.1.44 *Let E be a finite-dimensional vector space, and let F_1, F_2 be subspaces of E, then:*

$$\dim(F_1 + F_2) = \dim F_1 + \dim F_2 - \dim(F_1 \cap F_2),$$

where $F_1 + F_2 = \{u + v\ /\ u \in F_1,\ v \in F_2\}$ is the sum of subspaces and $F_1 \cap F_2$ is their intersection.

Exercise 5.1.45 *Consider the subsets of $\mathbb{R}^3$:*

$$E = \left\{(x, y, 0) \in \mathbb{R}^3\ /\ x,\ y \in \mathbb{R}\right\} \text{ and } F = \{(x, 0, x)\ /\ x \in \mathbb{R}\}.$$

1. *Show that E and F are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.*
2. *Calculate $\dim(E)$ and $\dim(F)$.*
3. *Determine $E \cap F$.*
4. *Is $\mathbb{R}^3 = E \oplus F$?*

Solution 5.1.46 *1. Show that E and F are vector subspaces of $\mathbb{R}^3$ over $\mathbb{R}$.*

- *To show that E is a vector subspace, we verify the following conditions:*

(a) Non-empty: The zero vector $(0,0,0) \in E$ (taking $x = 0, y = 0$).

(b) Closed under addition: Let $u = (x_1, y_1, 0)$ and $v = (x_2, y_2, 0)$ be in E. Then:

$$u + v = (x_1 + x_2, y_1 + y_2, 0) \in E.$$

(c) Closed under scalar multiplication: Let $u = (x, y, 0) \in E$ and $\lambda \in \mathbb{R}$. Then:

$$\lambda u = (\lambda x, \lambda y, 0) \in E.$$

Thus, E is a vector subspace.

- *For F :*

(a) Non-empty: The zero vector $(0,0,0) \in F$ (taking $x = 0$).

(b) Closed under addition: Let $u = (x_1, 0, x_1)$ and $v = (x_2, 0, x_2)$ be in F. Then:

$$u + v = (x_1 + x_2, 0, x_1 + x_2) \in F.$$

(c) Closed under scalar multiplication: Let $u = (x, 0, x) \in F$ and $\lambda \in \mathbb{R}$. Then:

$$\lambda u = (\lambda x, 0, \lambda x) \in F.$$

Thus, $F$is also a vector subspace.

2. Calculate $\dim(E)$, $\dim(F)$

(a) To find the dimension of E, we have:

$$\begin{aligned} E &= \{(x, y, 0) : x, y \in \mathbb{R}\} \\ &= \{x(1,0,0) + y(0,1,0) : x, y \in \mathbb{R}\} \\ &= span\{(1,0,0), (0,1,0)\}. \end{aligned}$$

The vectors $(1,0,0)$ and $(0,1,0)$ are linearly independent and therefore form a basis for E. Thus, we conclude that the dimension of E is: $\dim(E) = 2$.

(b) To find the dimension of F:The vector$(1,0,1)$ spans F since any vector in F can be expressed as $x(1,0,1)$ for some x. Thus, we have:

$$\dim(F) = 1.$$

3. To find $E \cap F$, we note that:

$$(x, y, z) \in E \cap F \Rightarrow (x, y, z) \in E \text{ and } (x, y, z) \in F.$$

This implies:

$$(x, y, z) \in E \Rightarrow z = 0, (x, y, z) \in F \Rightarrow y = 0 \text{ and } z = x.$$

Thus, the intersection is:

$$E \cap F = \{(0, 0, 0)\}.$$

4. The dimension of $E + F$ can be calculated using the formula:

$$\dim(E + F) = \dim(E) + \dim(F) - \dim(E \cap F).$$

Substituting the dimensions, we find:

$$\dim(E + F) = 2 + 1 - 0 = 3.$$

Since $\dim\left(\mathbb{R}^3\right) = 3$ *and* $\dim(E + F) = 3$*, we conclude that* $E + F = \mathbb{R}^3$*. Furthermore, since* $E + F = \mathbb{R}^3$ *and from Question 3 we have* $E \cap F = \{(0, 0, 0)\}$*, we conclude that* $\mathbb{R}^3 = E \oplus F$*.*

5.2 Linear applications

5.2.1 Definitions and examples

Definition 5.2.1 (linear map) *Let E and F be two vector spaces over a field $\mathbb{K}$.*

A map $f : E \to F$ is called linear if it satisfies both of the following conditions:

$$\begin{array}{l} \forall x,\ y \in E,\ f(x+y) = f(x) + f(y), \\ \forall x \in E,\ \forall \lambda \in \mathbb{K},\ f(\lambda x) = \lambda f(x), \end{array}$$

Equivalently,

$$\forall x,\ y \in E,\ \lambda \in \mathbb{K},\ f(\lambda x + y) = \lambda f(x) + f(y).$$

Remark 5.2.2 *The set of linear maps of E to F is denoted by $\mathcal{L}(E,\ F)$.*

Example 5.2.3 *The map f defined by*

$$\begin{array}{l} f : \mathbb{R}^3 \to \mathbb{R}^2, \\ (x,y,z) \to f(x,y,z) = (2x+y, y-z) \end{array},$$

is a linear map.

Indeed, let (x,y,z), $(\acute{x},\acute{y},\acute{z}) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} f\left[(x,y,z) + (\acute{x},\acute{y},\acute{z})\right] &= f\left(x+\acute{x}, y+\acute{y}, z+\acute{z}\right) \\ &= \left(2\left(x+\acute{x}\right) + \left(y+\acute{y}\right), \left(y+\acute{y}\right) - \left(z+\acute{z}\right)\right) \\ &= \left(2x+2\acute{x}+y+\acute{y}, y+\acute{y}-z-\acute{z}\right) \\ &= \left((2x+y) + \left(2\acute{x}+\acute{y}\right), (y-z) + \left(\acute{y}-\acute{z}\right)\right) \\ &= (2x+y, y-z) + \left(2\acute{x}+\acute{y}, \acute{y}-\acute{z}\right) \\ &= f(x,y,z) + f\left(\acute{x},\acute{y},\acute{z}\right), \end{aligned}$$

and

$$\begin{aligned} f(\lambda(x,y,z)) &= f(\lambda x, \lambda y, \lambda z) \\ &= (2\lambda x + \lambda y, \lambda y - \lambda z) \\ &= (\lambda(2x+y), \lambda(y-z)) \\ &= \lambda(2x+y, y-z) \\ &= \lambda f(x,y,z). \end{aligned}$$

Example 5.2.4 *The map* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *defined by*

$$f(x, y) = \left(x^2, x + y, 1\right)$$

is not linear.

Indeed,

$$f\left((1, 0) + (0, 0)\right) = f\left(1, 0\right) = \left(1, 1, 1\right),$$

whereas

$$f(1, 0) + f(0, 0) = (1, 1, 1) + (0, 0, 1) = (1, 1, 2)\,.$$

hence,

$$f\left((1, 0) + (0, 0)\right) \neq f(1, 0) + f(0, 0).$$

Proposition 5.2.5 *If f is a linear map from E to F, then :*

1.$f(0_E) = 0_F$.

2. $f(-x) = -f(x)$.

3. If V_1 is a subspace of E , then $f(V_1)$ is a subspace of F.

4. If W_1 is a subspace of F, then $f^{-1}(W_1)$ is a subspace of E.

5. The composition of two linear maps is a linear map.

Proposition 5.2.6 *Let E and F be vector spaces over K, and let $f,\ g \in \mathcal{L}\left(E,\ F\right)$. If E is finite-dimensional of dimension n and $\{e_1,\ e_2,\ ...,\ e_n\}$ is basis of E, then*

$$\forall k \in \{1, 2, ..., n\}\,,\ f(e_k) = g(e_k) \Leftrightarrow \forall x \in E,\ f(x) = g(x).$$

Proof. The implication ($\Leftarrow$) is obvious.

For ($\Rightarrow$), since $\{e_1,\ e_2,\ ...,\ e_n\}$ generates E, for any $x \in E$ there exist scalars $\lambda_1, \lambda_2, ..., \lambda_n \in \mathbb{K}$ such that

$$x = \sum_{i=1}^{n} \lambda_i e.$$

Since f and g are linear maps,

$$f\left(x\right) = \sum_{i=1}^{n} \lambda_i f\left(e_i\right),\ g\left(x\right) = \sum_{i=1}^{n} \lambda_i g\left(e_i\right).$$

If $f\left(e_i\right) = g\left(e_i\right)$ for all i, then $f\left(x\right) = g\left(x\right)$for all $x \in E$. ■

5.2.2 Linear maps and dimension

Let $f : E \to F$ be a linear map.

The kernel of a linear map

Definition 5.2.7 *The kernel (or null space) of* f*, denoted by* $\ker f$*, is the set of all vectors* $x \in E$ *such that* $f(x) = 0_F$ *(the zero vector of* F *):*

$$\ker f = \{x \in E \ \ / \ f(x) = 0_F\} = f^{-1}(\{0_F\})$$

The image of a linear map

Definition 5.2.8 *The image of* f*, denoted by* $\operatorname{Im} f$*, is the set of all vectors in* F *of the form* $f(x)$ *for some* $x \in E$*:*

$$\operatorname{Im} f = \{f(x) \ \ / \ x \in E\} = f(E)$$

Proposition 5.2.9 *Let* $f : E \to F$ *be a linear map. Then:*

1. $\ker f$ *is a subspace of* E*.*
2. $\operatorname{Im} f$ *is a subspace of* F*.*
3. f *is injective if and only if* $\ker f = \{0_E\}$*.*
4. f *is surjective if and only if* $\operatorname{Im} f = F$*.*

Example 5.2.10 *Consider the map* $f : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by*

$$f(x, y, z) = (x + y, z).$$

This map is not injective but is surjective.

- *Injectivity*

$$\begin{aligned} \ker f &= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ (x + y, z) = (0, 0)\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ x + y = 0, \ z = 0\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \ \ / \ y = -x, \ z = 0\} \\ &= \{(x, -x, 0) \ / \ x \in \mathbb{R}\} \end{aligned}$$

Since

$$(1, -1, 0) \in \ker f \Rightarrow \ker f \neq \{0_{\mathbb{R}^3}\}.$$

Hence, f *is not injective.*

- *Surjectivity.*

$$\begin{aligned} \operatorname{Im} f &= \left\{(x + y, z) \ / \ (x, y, z) \in \mathbb{R}^3\right\} \\ &= \{x(1, 0) + y(1, 0) + z(0, 1) \ / \ x, y, z \in \mathbb{R}\}. \end{aligned}$$

Thus,

$$\operatorname{Im} f = span\{(1,0),\ (0,1)\} = \mathbb{R}^2,$$

and f is surjective.

Proposition 5.2.11 *Let $f: E \to F$ be a linear map, with E of finite dimension. Then:*

$$\dim E = \dim \ker f + \dim \operatorname{Im} f$$

The rank of a linear map

Definition 5.2.12 *The rank of a linear map f is the dimension of its image :*

$$rankf = \dim \operatorname{Im} f$$

Example 5.2.13 *Find $\ker f$, $\operatorname{Im} f$ and $rankf$ for the map $f: \mathbb{R}^4 \to \mathbb{R}^3$ defined by*

$$f(x,y,z,t) = (x-y, z+t, x-y+z)$$

Kernel

$$\ker f = \left\{(x,y,z,t) \in \mathbb{R}^4 \ / \ (x-y, z+t, x-y+z) = (0,0,0)\right\}$$

From $x-y=0$, we get $x=y$.

From $x-y+z=0$, we get $z=0$, hence $t=0$.

Thus,

$$\ker f = \{(x,x,0,0) \ / \ x \in \mathbb{R}\} = span\{(1,1,0,0)\}.$$

Image

$$\begin{aligned} \operatorname{Im} f &= \{(x-y, z+t, x-y+z) \ / \ x,y,z,t \in \mathbb{R}\} \\ &= \{(x-y).(1,0,1) + t.(0,1,0) + z(0,1,1) \ / \ x,y,z,t \in \mathbb{R}\} \\ &= span\{(1,0,1),\ (0,1,0),\ z(0,1,1)\}. \end{aligned}$$

To check linear independence, let $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathbb{R}$:

$$\begin{aligned} &\lambda_1(1,0,1) + \lambda_2(0,1,0) + \lambda_3(0,1,1) = (0,0,0) \\ &\qquad \Rightarrow (\lambda_1, \lambda_2 + \lambda_3, \lambda_1 + \lambda_3) = (0,0,0) \\ &\qquad \Rightarrow \lambda_1 = \lambda_1 = \lambda_1 = 0. \end{aligned}$$

Hence, the vectors are linearly independent and form a basis of $\operatorname{Im} f$.

$$rankf = \dim \operatorname{Im} f = 3.$$