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L1

Series n°4 - Maths1

Exercise 1
A law $*$ is definedin the interval $E =]-1, +1[$ by : $x * y = \frac{x+y}{1+xy}$

1. Show that the operation is internal in E.
2. Show that $(E, *)$ is a commutatif group.

Exercise 2
Let $(\mathbb{C}, +, .)$ be a ring and let $(\mathbb{R}, +, .)$ be a body, we consider the subset H of $\mathbb{C}$ and the subset K of $\mathbb{R}$, where

$$H = \left\{a + bi/\ a, b \in \mathbb{Z}, i^2 = -1\right\}$$

$$K = \left\{a + b\sqrt{2}/\ a, b \in \mathbb{Q}\right\}$$

1. Show that $(H, +, .)$ is a sub-ring of $\mathbb{C}$.

2 Show that $(K, +, .)$ is a sub-body of $\mathbb{R}$.

Exercise 3
We consider the sets $E = \{(x, y) \in \mathbb{R}^2; y = 2x\}$ and $F = \{(x, y) \in \mathbb{R}^2/\ y = x\}$.

1. Show that E and F are vector subspaces of $\mathbb{R}^2$.
2. Determine a basis of E and F.
3. Determine $E \cap F$.

Exercise 4
Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be an application defined fo all $(x, y, z) \in \mathbb{R}^3$ by :

$$f(x, y, z) = (-2x + y + z, x - 2y + z)$$

1. Prove that f is a linear application.
2. Determine a basis of $ker(f)$, deduce $dim(Img(f))$.
3. Determine a basis of $dim(Img(f))$.

Exercise 5 (It can be solved in Course)
Let $f : \mathbb{R}^4 \to \mathbb{R}^4$ defined for all $(x, y, z, t) \in \mathbb{R}^4$ by

$$f(x, y, z, t) = (x - 2y, x - 2y, 0, x - y - z - t)$$

1. Prove that f is a linear application.

2. Determine $ker(f)$ and $Img(f)$.

3. Is $\ker(f) \oplus Img(f) = \mathbb{R}^4$?

Exercise 6 (It can be solved in Course)
Let $f : R^3 - \to R^2$ be a map defined by $f(x, y, z) = (x + y - z, x - y + 2z)$

1. Determine $\ker(f)$ et $Img(f)$.

2. Is f injective? surjective?

Correction

Exercise 1

A law $*$ is defined in the interval $E =]-1, +1[$ by : $x * y = \frac{x+y}{1+xy}$

1. Show that the operation is internal in E

To prove that $*$ is an internal operation on E, we need to verify that for any $x, y \in E$, the result $x * y \in E$.

We have $\begin{cases} x \in]-1,+1[\Rightarrow |x| < 1 \\ y \in]-1,+1[\Rightarrow |y| < 1 \end{cases}$, then $|xy| = |x|\,|y| < 1$

So $-1 < xy < 1$, and therefore $0 < 1 + xy < 2$

We have

$$\begin{cases} -1 < x \Rightarrow x + 1 > 0 \\ -1 < y \Rightarrow 1 + y > 0 \end{cases} \Rightarrow (1+x)(1+y) > 0$$

Since $1 + xy > 0$, we get $\frac{(1+x)(1+y)}{1+xy} > 0 \Rightarrow 1 + \frac{x+y}{1+xy} > 0 \quad ((1+x)(1+y) = 1 + xy + x + y)$

$\Rightarrow \frac{x+y}{1+xy} > -1$..........(1)

In the second cases, we have

$$\begin{cases} x < 1 \Rightarrow 1 - x > 0 \\ y < 1 \Rightarrow 1 - y > 0 \end{cases} \Rightarrow (1-x)(1-y) > 0$$

Since $1 + xy > 0$, we get $\frac{(1-x)(1-y)}{1+xy} > 0 \Rightarrow 1 - \frac{x+y}{1+xy} > 0 \quad ((1-x)(1-y) = 1 + xy - x - y)$

$\Rightarrow \frac{x+y}{1+xy} < 1$..............(2)

From (1) and (2), we obtain $-1 < x * y < 1$.

2. Show that $(E, *)$ is a commutative group

a. Commutativity and associativity : it's eazy to verify the commutativity and the assiciativty of the operation $*$ in E.

b. Identity (neutral) element :

$\exists e \in E, \forall x \in E : e * x = x$

Let $x \in E$, we have

$e * x = x \Rightarrow \frac{e+x}{1+ex} = e$

$\Rightarrow e = 0 \in E$

Thus, 0 is the identity element.

3. Inverse (symmetrical) element :

$\forall x \in E, \exists x' \in E : x * x' = e$

We have $x * x' = e \Rightarrow \frac{x+x'}{1+xx'} = 0$

$\Rightarrow x' = -x$

For each $x \in E$, its inverse under $*$ is $-x$.

Therefore, $(E, *)$ is a commutative group.

Exercise 2

1) We show that $(H, +, .)$ is a sub-ring.

a) $H \neq \emptyset$, because $0 = 0 + 0i$ where $0 \in \mathbb{Z}$.

b) $\forall x, y \in H : x - y \in H$

Let $x, y \in H$

$x \in H \Rightarrow x = a + bi \quad /a, b \in \mathbb{Z}$

$y \in H \Rightarrow y = c + di \quad /c, d \in \mathbb{Z}$
Then, $x - y = a - c + (b - d)\, i$ where $a - c \in \mathbb{Z}$, $b - d \in \mathbb{Z}$.
Hence $x - y \in H$.
c) $\forall x, y \in H : x.y \in H$
We have $x.y = (a + bi)(c + di) = ac - bd + (ad - bc)\, i$ where $ac - bd \in \mathbb{Z}$, $ad - bc \in \mathbb{Z}$.
Thus, H is a sub-ring of $\mathbb{C}$.
2) Show that $(K, +, .)$ is a sub-body.
a) $K \neq \emptyset$, because $0 = 0 + 0\sqrt{2}$ where $0 \in \mathbb{Q}$.
b) $\forall x, y \in K : x - y \in K$
Let $x, y \in K$
$x \in K \Rightarrow x = a + b\sqrt{2} \quad /a, b \in \mathbb{Q}$
$y \in K \Rightarrow y = c + d\sqrt{2} \quad /c, d \in \mathbb{Q}$
Then, $x - y = a - c + (b - d)\sqrt{2}$ where $a - c \in \mathbb{Q}$, $b - d \in \mathbb{Q}$.
Hence $x - y \in K$.
c) $\forall x \in K, \forall y \in K - \{0\} : x.y^{-1} \in K$
If $y \neq 0$, then $y^{-1} = \frac{1}{y} = \frac{1}{c+d\sqrt{2}} = \frac{c-d\sqrt{2}}{c^2-2d^2} = \frac{c}{c^2-2d^2} - \frac{d}{c^2-2d^2}\sqrt{2}$ where $\frac{c}{c^2-2d^2} \in \mathbb{Q}$, $\frac{d}{c^2-2d^2} \in \mathbb{Q}$.
So $y^{-1} \in K$.
We have $x.y^{-1} = \left(a + b\sqrt{2}\right)\left(\frac{c}{c^2-2d^2} - \frac{d}{c^2-2d^2}\sqrt{2}\right) = \left(\frac{ac-2bd}{c^2-2d^2}\right) + \left(\frac{cb-ad}{c^2-2d^2}\right)\sqrt{2}$
Hence $x.y^{-1} \in K$.
Thus, K is a sub-body of $\mathbb{R}$.

Exercice 3

1) We show that E is a vector subspace of $\mathbb{R}^2$.
a) $(0, 0)$ is an element of E, so $E \neq \emptyset$.
b) Let $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ be two elements of E.
$X + Y = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.
since X and Y belong to E, $y_1 = 2x_1$and $y_2 = 2x_2$ and therefore,
$y_1 + y_2 = (2x_1) + (2x_2) = 2(x_1 + x_2)$.
So $X + Y$ is also an element of E.
c) $\forall \alpha \in \mathbb{R}, \forall X = X = (x, y) \in E$, we have $\alpha X = \alpha(x, y) = (\alpha x, \alpha y)$.
since X belongs to E, $y = 2x$ and therefore, $\alpha y = \alpha\,(2x) = 2\,(\alpha x)$
So αX is also an element of E.
Thus E is a vector subspace of $\mathbb{R}^2$
2) We show that F is a vector subspace of $\mathbb{R}^2$.
a) $(0, 0)$ is an element of F thus $F \neq \emptyset$.
b) Let $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ be two elements of F and α, β any two reals
$\alpha X + \beta Y = \alpha(x_1, y_1) + \beta(x_2, y_2) = (\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) = (\alpha x_1 + \beta x_2, \alpha x_1 + \beta x_2)$.
So $\alpha X + \beta Y$ is also an element of F and F is a vector subspace of $\mathbb{R}^2$.
3) Determine a basis of E.
$E = \{(x, y) \in \mathbb{R}^2; y = 2x\}$
$= \{(x, 2x) \ / \ x \in \mathbb{R}\}$
$= \{x(1, 2) \ / \ x \in \mathbb{R}\}$
$= Vect\,((1, 2))$
Since $(1, 2) \neq (0, 0)$, $\{(1, 2)\}$ is a basis of E.
4) Determine a basis of F
$F = \{(x, y) \in \mathbb{R}^2 / \ y = x\}$
$= \{(x, x) \ /x \in \mathbb{R}\}$

$= \{x(1,1)\ /x \in \mathbb{R}\}$
$= Vect\,((1,1))$
Since $(1,1) \neq (0,0)$, $\{(1,1)\}$ is a basis of F.
5) $E \cap F = \{(x,y) \in \mathbb{R}^2; (x,y) \in E \text{ and } (x,y) \in F\} = \{(x,y) \in \mathbb{R}^2; y = 2x \text{ and } y = x\}$
$E \cap F = \{(x,y) \in \mathbb{R}^2; y = 2x = x\} = \{(x,y) \in \mathbb{R}^2; x = 0 \text{ and } y = 0\}$
$E \cap F = \{(0,0)\}$.
Exercice 4.
1) $\forall\,(x,y,z)\,,(x',y',z') \in \mathbb{R}^3 : f\left((x,y,z) + (x',y',z')\right) = f\,(x,y,z) + f\,(x',y',z')$
Let $(x,y,z)\,,(x',y',z') \in \mathbb{R}^3$, we have

$$\begin{aligned} f\left((x,y,z)+(x',y',z')\right) &= f\,(x+x',y+y',z+z') \\ &= \left(-2\,(x+x')+(y+y')+(z+z')\,,(x+x')-2\,(y+y')+(z+z')\right) \\ &= \left((-2x+y+z)+(-2x'+y'+z')\,,(x-2y+z)+(x'-2y'+z')\right) \\ &= (-2x+y+z,x-2y+z)+(-2x'+y'+z',x'-2y'+z') \\ &= f\,(x,y,z)+f\,(x',y',z') \end{aligned}$$

$\forall \alpha \in \mathbb{R}, \forall (x,y,z) \in \mathbb{R}^3 : f\left(\alpha(x,y,z)\right) = \alpha f(x,y,z)$
Let $(x,y,z) \in \mathbb{R}^3$, we have
$f\left(\alpha(x,y,z)\right) = f\left(\alpha x, \alpha y, \alpha z\right) = (-2\alpha x + \alpha y + \alpha z, \alpha x - 2\alpha y + \alpha z)$
$= \left(\alpha\,(-2x+y+z)\,, \alpha\,(x-2y+z)\right)$
$= \alpha f(x,y,z)$.
So f is a linear map.
2) $ker(f) = \{(x,y,z) \in \mathbb{R}^3/\ f\,(x,y,z) = 0_{\mathbb{R}^2}\}$
$= \{(x,y,z) \in \mathbb{R}^3/(-2x+y+z, x-2y+z) = (0,0)\}$
$= \{(x,y,z) \in \mathbb{R}^3/-2x+y+z = 0,\ x-2y+z = 0\}$
We have $\begin{cases} -2x+y+z=0 & (1) \\ x-2y+z=0 & (2) \end{cases}$
$2\,(2)+(1) : \quad -3y+3z = 0 \Rightarrow y = z$
From (2), we get $x = z$
So, $ker(f) = \{(z,z,z)\,/\ z \in \mathbb{R}\}$
$= \{z\,(1,1,1)\,/\ z \in \mathbb{R}\}$
$= Vect\,((1,1,1))$
then $dim(\ker(f)) = 1$.
According to the rank theorem
$dim(\ker(f)) + dim(Img(f)) = dim(\mathbb{R}^3) \Leftrightarrow dim(Img(f)) = dim(\mathbb{R}^3) - dim(\ker(f))$
$\Leftrightarrow dim(Img(f)) = 3 - 1 = 2$
3) Since $dim(Img(f)) = dim(\mathbb{R}^2)$, then $Img(f) = \mathbb{R}^2$, $\{(1,0)\,,(0,1)\}$ is a basis of $Img(f)$.
Exercice 5.
1) It is easily to verify that f is a linear application.
2) $ker(f) = \{(x,y,z,t) \in \mathbb{R}^4/\ f\,(x,y,z,t) = 0_{\mathbb{R}^4}\}$
$= \{(x,y,z,t) \in \mathbb{R}^4/\ (x-2y, x-2y, 0, x-y-z-t) = (0,0,0,0)\}$
$= \{(x,y,z,t) \in \mathbb{R}^4/\ x-2y = 0,\ x-y-z-t = 0\}$
$= \{(x,y,z,t) \in \mathbb{R}^4/\ x = 2y,\ t = y-z\}$
$= \{(2y,y,z,y-z)\ \ /\ y,z \in \mathbb{R}^2\}$
$= \{y\,(2,1,0,1) + z\,(0,0,1,-1)\ \ /\ y,z \in \mathbb{R}^2\}$

$= Vect\left((2,1,0,1),(0,0,1,-1)\right)$

Since $(2,1,0,1),(0,0,1,-1)$ are linearly independent, $\{(2,1,0,1),(0,0,1,-1)\}$ is a basis of $ker(f)$, then $\dim(ker(f)) = 2$.

$$\begin{aligned}
\operatorname{Im} g(f) &= \{f(x,y,z,t) \ \ / \ (x,y,z,t) \in \mathbb{R}^4\} \\
&= \{\ (x-2y, x-2y, 0, x-y-z-t) \ \ / \ (x,y,z,t) \in \mathbb{R}^4\} \\
&= \{(x,x,0,x) + (-2y,-2y,0,-y) + (0,0,0,-z-t) \ / \ (x,y,z,t) \in \mathbb{R}^4\} \\
&= \{x(1,1,0,1) + y(-2,-2,0,-1) + (z+t)(0,0,0,-1) \ / \ (x,y,z,t) \in \mathbb{R}^4\} \\
&= Vect\left((1,1,0,1),(-2,-2,0,-1),(0,0,0,-1)\right)
\end{aligned}$$

but the vectors $(1,1,0,1),(-2,-2,0,-1),(0,0,0,-1)$ are not linearly independent.

Since $(1,1,0,1),(0,0,0,-1)$ are linearly independent then, $\dim(\operatorname{Im} g(f)) = 2$.

3) Is $\ker(f) \oplus Img(f) = \mathbb{R}^4$?

We have $\ker(f) \oplus Img(f) = \mathbb{R}^4 \Leftrightarrow \{(2,1,0,1),(0,0,1,-1),(1,1,0,1),(0,0,0,-1)\}$ is a basis of $\mathbb{R}^4$.

Since $(2,1,0,1),(0,0,1,-1),(1,1,0,1),(0,0,0,-1)$ are linearly independent then, $\{(2,1,0,1),(0,0,1,-1),(1,1,0,1),(0,0,0,-1)\}$ is a basis of $\mathbb{R}^4$, so $\ker(f) \oplus Img(f) = \mathbb{R}^4$.

Exercice 6

1)
$$\begin{aligned}
ker(f) &= \{(x,y,z) \in \mathbb{R}^3 / \ f(x,y,z) = 0_{\mathbb{R}^2}\} \\
&= \{(x,y,z) \in \mathbb{R}^3 / \ (x+y-z, x-y+2z) = (0,0)\} \\
&= \{(x,y,z) \in \mathbb{R}^3 / \ x+y-z = 0,\ x-y+2z = 0\}
\end{aligned}$$

We have $\begin{cases} x+y-z=0 & (1) \\ x-y+2z=0 & (2) \end{cases}$

$(2)+(1): \quad 2x+z=0 \Rightarrow z=-2x$

From (1), we get $y=-3x$

So,
$$\begin{aligned}
ker(f) &= \{(x,-3x,-2x) \ / \ x \in \mathbb{R}\} \\
&= \{x(1,-3,-2) \ / \ x \in \mathbb{R}\} \\
&= Vect\left((1,-3,-2)\right)
\end{aligned}$$

then f is not injective $(ker(f) \neq \{0_{\mathbb{R}^3}\})$

$$\begin{aligned}
\operatorname{Im} g(f) &= \{f(x,y,z) \ \ / \ (x,y,z) \in \mathbb{R}^3\} \\
&= \{\ (x+y-z, x-y+2z) \ \ / \ (x,y,z) \in \mathbb{R}^3\} \\
&= \{(x,x) + (y,-y) + (-z,2z) \ / \ (x,y,z) \in \mathbb{R}^3\} \\
&= \{x(1,1) + y(1,-1) + z(-1,2) \ / \ (x,y,z) \in \mathbb{R}^3\} \\
&= Vect\left((1,1),(1,-1),(-1,2)\right)
\end{aligned}$$

but the vectors $(1,1),(1,-1),(-1,2)$ are not linearly independent.

Since $(1,1),(1,-1)$ are linearly independent then, $\dim(\operatorname{Im} g(f)) = 2 = \dim \mathbb{R}^2$

So, $\operatorname{Im} g(f) = \mathbb{R}^2$, thus f is surjective.